Infinity and Continuity

The mutual dependence and distinctness of *multiplicity* and *wholeness*

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Contents

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Practicing mathematicians, consciously or not,
subscribe to some philosophy of mathematics
(if unstudied, it is usually inconsistent)
(Monk, 1970:707)

1 Introduction

Although every single academic discipline employs *concepts* an explicit account of the *nature* of concept-formation is almost never encountered – in general the concept of a concept is absent. Normally key (or: supposedly *basic*) concepts are *defined* straightaway. Yet various disciplines did realize that definitions employ *terms* and that one cannot simply continue to define these terms without ending in a *regressus in infinitum*. Concept-formation and definition therefore ultimately rest upon *terms* which are *not defined* and cannot be defined.

How does one *know* these indefinable (primitive) terms? This is an epistemological issue which is rooted in *philosophical assumptions* about the world in which we live – and therefore involve *ontological commitments*.

Within the history of philosophy the underlying problem concerns the relation between *unity* and *diversity* and it centers in the question regarding the *coherence of irreducibiles*.

Greek mathematics commenced with an approach evidenced in the Pythagorean statement: "everything is number." The discovery of incommensurability by Hippasos of Metapont (about 450 B.C. – cf. Von Fritz, 1945) generated an emphasis on a new approach which attempted to consider our intuition of space as being more basic than that of number. Because intuition can grasp continuity all at once "Greek mathematics and philosophy were inclined to consider continuity to be the simpler concept" (Fraenkel *et al*., 1973:213). During the later part of the 19th century mathematics once again reverted to an arithmeticistic perspective – a process initiated by Bolzano and carried through by Weierstrass, Dedekind, and Cantor.

What is amazing about this course of events is that what seemed to have been solved twice still turned out to form the center of the problems involved, namely the discovery of irrationality (incommensurability) and the difficulties surrounding the theory of functions in Germany and France. Fraenkel et al. remark:

> Though the arguments have changed, the gap between discrete and continuous is again the weak spot – an eternal point of least resistance and at the same time of overwhelming scientific importance in mathematics, philosophy, and even physics (1973:212-213).

It is indeed amazing that the entire history of mathematics only explored the following two options: either reduce space to number or reduce number to space.[1] The obvious *third* option was never examined: accept both the uniqueness (i.e., *irreducibility*) of number and space *and* their mutual interconnectedness!

In order to investigate some of the implications entailed in such a third alternative we start by looking at the implicit assumptions of modern axiomatic set theory.

2 Probing some foundational designs

Although Euclid already employed the axiomatic method it received its modern rigour from Hilbert's work on the foundations of geometry (*Grundlagen der Geometrie*, 1899). In this work Hilbert abstracts from the contents of his axioms, based upon three *undefined* terms: "point," "lies on," and "line."

2.1 *The impasse of logicism*

Frege, Dedekind and Russell advanced a *logicistic approach* to the foundations of mathematics. Dedekind started from an actual infinity of "objects" within my "Gedankenwelt" (translated by Rucker with the descriptive term "mindscape" – Rucker, 1982:47). Russell defines *number* with the aid of his supposedly *purely logical* concept of *class*. The logical concept, he claims, enables the reduction of mathematics to logic. For example, the number "2" is "defined" in the following way: "1 + 1 is the number of a class *w* which is the logical sum of two classes *u* and *v* which have no common terms and have each only one term. The chief point to be observed is, that logical addition of numbers is the fundamental notion, while arithmetical addition of numbers is wholly subsequent" (Russell, 1956:119).

The irony, however, is that Russell already had to use the *meaning of number* in order to distinguish between different ("logical") classes. After all, he speaks about the sum of "two" classes where each of them contains "one" element. This presupposes an insight into the *numerical meaning* of the numbers "1" and "2"! Consequently, the number "2," which had to appear as the *result* of "logical addition," is *presupposed* by it! In his discussion of *number and*

1 It should be kept in mind that Frege eventually showed an affinity with the latter position. Dummett refers to Frege's own "eventual expedient of reducing arithmetic to geometry" (1995:319). With regard to a preference for geometrical intuitions Wang writes: "An alternative course would be to consider our geometrical intuitions, as Plato and Bernays (and, I understand, also Frege in his later years) apparently preferred" (1988:203).

the concept of class Cassirer displays a clear understanding of this circularity (1953:44 ff.).[1]

Although Dedekind asserts that the *idea of infinity* should form part of the *logical foundation* of mathematics, it soon turned out that the meaning of infinity *precedes* logic. Hilbert points out that in contrast to the early attempts of Frege and Dedekind he is convinced that as a precondition for the possibility of scientific knowledge certain intuitive representations and insights are indispensible and that logic alone is not sufficient.[2] Fraenkel *et al.* also affirm: "It seems, then, that the only really serious drawback in the Frege-Russell thesis is the doubtful status of InfAx,[3] according to the interpretation intended by them" (1973:186). Myhill mentions the fact that the axioms of *Principia* do not determine how many individuals there are: "the axiom of infinity, which is needed as a hypothesis for the development of mathematics in that system, is neither provable nor refutable therein, *i.e.*, is undecidable" (Myhill, 1952:182).

In anticipation of our subsequent analysis of the inter-modal coherence between aspects we have to point here at the foundational role of the meaning of number with respect to that of the meaning of analysis. Every attempt to deduce the meaning of number from the meaning of analysis (or: logic) is faced with a *vicious circle*. Cassirer is also quite explicit in this regard. He claims that a critical analysis of knowledge, in order to side-step a regressus *in infinitum*, has to accept certain basic functions which are not capable of being "deduced" and which are not in need of a deduction.[4] David Hilbert also points at this "catch 22" entailed in the logicist attempt to deduce the meaning of number from that of the logical-analytical mode.

In his *Gesammelte Abhandlungen* Hilbert writes: "Only when we analyze attentively do we realize that in presenting the laws of logic we already had to employ certain arithmetical basic concepts, for example the concept of a set and partially also the concept of number, particularly as cardinal number [Anzahl]. Here we end up in a vicious circle and in order to avoid paradoxes it is necessary to come to a partially simultaneous development of the laws of logic and arithmetic" (1970:199).

1 Singh also points out that Russell's attempt makes him a victim of the "vicious circle principle" (1985:76).

2 "Im Gegensatz zu den früheren Bestrebungen von Frege und Dedekind erlangen wir die Überzeugung, daß als Vorbedingung für die Möglichkeit wissenschaftlicher Erkenntnis gewisse anshauliche Vorstellungen und Einsichten unentbehrlich sind und die Logik allein nicht ausreicht" – Hilbert, 1925:190.

3 InfAx = *Axiom of Infinity*.

4 "Denn die kritische Analyse der Erkenntnis wird, wenn man nicht einen regressus in infinitum annehmen will, immer bei gewissen Urfunktionen Halt machen müssen, die einer eigentlichen 'Ableitung' weder fähig noch bedurftig sind" (1957:73).

2.2 *Contradiction and the meaning of analysis*

The initial stage of mathematical set theory, as developed by Georg Cantor (between 1874 and 1899), got stuck in contradictions. By 1895 Cantor himself discovered that his set theory contains anomalies. Cantor proved e.g. the proposition that for every set *A* of ordinal numbers an ordinal number exists which is greater than every ordinal number contained in the set. Consider however the set *W* of *all* ordinal numbers. Since this set is a set of all ordinal numbers, the foregoing proposition implies that an ordinal number exists which is greater than every ordinal number contained in *W* – but this is contradictory, since the set *W* supposedly already contains *all* ordinal numbers. A similar contradiction holds with regard to Cantor's cardinal numbers (cf. Meschkowski, 1967:144-145 and Singh, 1985:73).

The ensuing *axiomatization* of set theory, for example that of Zermelo-Fraenkel, proceeds on the basis of (i) *first order predicate calculus*[1] and (ii) it introduces as undefined term the specific set-theoretical primitive binary predicate $\in$ which is called the *membership relation* (Fraenkel et al., 1973:23).[2] This approach follows a general pattern: an axiomatic theory (axiomatic theories of logic excluded) "is constructed by adding to a certain *basic discipline* – usually some system of logic (with or without a set theory) but sometimes also a system of arithmetic – new terms and axioms, the *specific* undefined terms and axioms under consideration" (Fraenkel et al., 1973:18).

The first order predicate calculus assumed by ZF contains a set of *connectives* enabling the expression of *negation* ($\neg$), *conjunction* ($\wedge$), *disjunction* ($\vee$), *conditional* ($\rightarrow$), *biconditional* ($\leftrightarrow$) and the two quantifiers: the *universal quantifier* ($\forall$ / for all) and the *existential quantifier* ($\exists$ / there exist). This means that the underlying logic provides ZF Set Theory with the following primitive symbols: *connectives*, *quantifiers* and, in addition, also *variables*.

These connectives testify to the fact that the analytical mode is *normed* in the sense that it makes possible *concept-formation* and *argumentation* which may or may not conform to normative logical principles [such as the principle of identity, (non-)contradiction, the excluded middle, and so on]. Negation makes it possible to assess *illogical* thinking and *contradictions* – a statement and its negation cannot both be true in the same context. Negation as a connective presupposes the logical principles of identity and (non-) contradiction – in what is analyzable A is A and A is not non-A. Thus analysis presupposes *unity* and *multiplicity*. First of all this relates to *arithmetical phenomena.*

1 Van Heijenoort remarks that "an axiomatization of set theory is usually embedded in a logical calculus, and it is Weyl's and Skolem's approach to the formulation of the axiom of separation that is generally adopted" (1967:285).

2 Note that within Zermelo-Fraenkel Set Theory (ZF) the terms "set" and "element" are synonymous (1973:24), implying that this theory will avoid the phrase "*x* is an element of *y*" (1973:23, note 2). Their terminology, in terms of the membership relation, is such that "$x \in y$" is read as "*x* is a member of *y*" or as "*x* belongs to *y*" (which is synonymous with "*x* is contained in *y*" / "*y* contains *x* (as a member)" (1973:23).

Therefore the meaning of the logical principle of identity and that of non-contradiction *analogically reflect* this basic arithmetical meaning of unity and multiplicity. Whatever is given as a discrete unity (as being distinct) is identical to itself (the basis of the principle of identity) and is different from whatever it is not (the basis of the principle of non-contradiction) (cf. Strauss, 1991). In themselves these two principles therefore (in a positive and negative way) reveal the *coherence* between two *irreducible* modal aspects of reality, namely the logical and the numerical modes.

In passing we may mention that arithmetical addition differs from a *logical synthesis* – something understood by Immanuel Kant. Where he argues for the *synthetic* nature of mathematical judgments in his *Critique of Pure Reason* (CPR), he clearly realizes that pure logical addition (a merely logical synthesis) cannot give rise to a *new number* (cf. CPR, 1787:15 where he considers the proposition that 7+5=12).

In a different way Frege made the same point: entitary directed abstraction can only proceed to more abstract entities, but never to any number as such. The logical addition of 'ones' or 'twos' cannot but end with the repeated identification of another number of the same kind: having identified a 'two' and another 'two' still provides us only with the 'abstract' notion of 'twoness'.

The presence of *variables* within the first order predicate calculus in yet another way also highlights the original meaning of number – but this time in coherence with the *kinematical* and the *physical* aspects. Physical changes presuppose constancy. Constancy as a term actually finds its "modal seat" within the *kinematical aspect* of *pure movement* – evidenced in Einstein's special theory of relativity where the velocity of light is postulated as being *constant* in a vacuum. The original meaning (modal seat) of *change* (variability) is found in the *physical aspect* where *energy-operation causes* certain *effects* (changes) to occur.

Remark: The conditional as connective analogically reflects the physical cause-effect relation – if *this* then one can conclude to *that*. Weyl (1966:32) touches upon the relation between the (physical) relationship between cause and effect on the one hand and ground and conclusion on the other without realizing that a basic concept of logic is at stake which analogically reflects the coherence between the logical and the physical aspects of reality – the relation between logical grounds and conclusions reflects the original physical relation of cause and effect.

He does realize that the logical relation may have its foundation in an applicable essential law, a "causal connection" or "an empirical regularity,"but nontheless claims that the "sign →"expresses a purely logical conclusion ('Folge')" without realizing that the inevitability to employ this term 'Folge' demonstrates the undeniable analogical link between the logical and the physical aspect of reality.

In the footsteps of Plato and Galileo it was clear to modern physics and to Einstein that *changes* could only be detected on the basis of *constancy*, explaining the close connection of *constants* and *variables* also in logic.[1]

Furthermore, the mere fact that both these terms appear in the *plural* makes it plain that the practice in logic to employ the terms *constants* and *variables* at once also presupposes the original quantitative *meaning* of number.

2.3 *Tacit assumptions of axiomatic set theory*

The first four axioms of ZF do not guarantee the existence of a set (or an object) at all (cf. Fraenkel et al., 1973:39, note 2).[2] In addition to these four Axioms the *Axiom of Subsets* (v) and the *Axiom of Infinity* (vi) are introduced.[3] The *Axiom of Infinity* asserts explicitly that some set exists (cf. Fraenkel et al., 1973:39, note 2).[4]

Let us now look at some of the tacit assumptions of axiomatic set theory.

In the absence of a foundational *ontological* consideration of the interrelationships between the numerical, the spatial, the kinematical, the physical and the analytical aspects of reality an extremely fundamental *circulus vitiosus* is concealed. Set theory is appreciated as a purely *arithmetical* theory (as intended by Cantor and most of his successors). Yet, in order to construct an *axiomatic foundation* for set theory, as we have seen, the aid of an underlying *logic* is required. Does this mean that logic in itself can provide a *sufficient* foundation for set theory (or: mathematics)? Whereas Russell claims that logic and mathematics are identical [1956:(v)] and made an attempt to derive the number concept from the logical class concept, we have seen that this entire procedure begs the question. The claims of logicism are untenable because both the logical class concept as well as the Axiom of Infinity make an appeal to the *basic meaning of number*. In other words, the very *meaning* of analysis (i.e, the logical mode of reality) presupposes the *meaning* of number and therefore cannot serve as a foundation for it.

1 We shall see that Weierstrass tried to eliminate the notion of variables through the postulation of a static infinite domain.

2 These Axioms are (Fraenkel et al., 1973:27-35): (i) the Axiom of Extensionality (if $x \subseteq y$ and $y \subseteq x$, then $x = y$); (ii) the Axiom of Pairing [for any two elements *a* and *b* there exists the set *y* which contains just *a* and *b* (i.e., *a* and *b* and no different member)]; (iii) the Axiom of Union / Sumset (for any set *a* there exists the set whose members are just the members of the members of *a*); (iv) the Axiom of Powerset (for any set *a* there exists the set whose members are just all the subsets of *a*).

3 The former (also known as the Axiom of *Separation*) states that for any set *a* and any condition $\mathscr{B}(x)$ on *x* there exists the set that contains just those members *x* of *a* which fulfil the condition $\mathscr{B}(x)$, while the latter (the Axiom of Infinity) secures the existence of infinite sets by *postulating* them (cf. Fraenkel et al., 1973:46). We shall presently return to the *form* of the InfAx.

4 For the moment we leave aside the *Axiom Schema of Replacement/Substitution*, the *Axiom of Choice* and the *Axiom Schema of Foundation*.

> The pretended foundation of set theory in logic therefore side-steps the crucial issue: in order to provide an axiomatic foundation for an analysis of the meaning of number an underlying logic is required which in itself presupposes this very meaning of number!

We have pointed out that this already follows from the inevitable presence of *quantifiers* and a *multiplicity* of (*constants* and) *variables* assumed in first order predicate calculus. The intuition of *multiplicity* is made possible by the unique quantitative meaning of the numerical aspect – first of all accounted for in the mastery of the *natural numbers* and in the fact that *succession* is also inherent to our understanding of natural numbers (the best known mathematical "application"of this order of succession is found in [mathematical/complete] *induction*).[1] Already in 1922 Skolem had a firm grip on these issues:

> Those engaged in doing set theory are normally convinced that the concept of an integer ought to be defined and that complete induction must be proved. Yet it is clear that one cannot define or provide an endless foundation; sooner or later one encounters what is indefinable or unprovable. Then the only option is to ensure that the first starting points are immediately clear, natural and beyond doubt. The concept of an integer and the inferences by induction meet this condition, but it is definitely not met by the set theoretic axioms such as those of Zermelo or similer ones. If one wishes to derive the former concepts from the latter, then the set theoretic concepts ought to be simpler and employing them then ought to be more certain than working with complete induction – but this contradict the real state of affairs totally (1979:70).[2]

3 The primacy of natural numbers and their succession (induction)

The intuition of one, another one, and so on generates the most basic meaning of infinity – literally without an end, endlessly, infinitely. The awareness of a *multiplicity* is at least accompanied by an awareness of *succession*. Kant realized that succession differs from causation: the day succeeds the night and the

1 According to Freudenthal, Dedekind was perhaps the first one (cf. Dedekind, 1887, par. 59, 80) to call the conclusion from n to $n + 1$ complete induction ("vollständige Induktion"). Neither Bernoulli nor Pascal is the founder of this principle. Its discovery must be credited to Francesco Maurolico (1494-1575) (cf. Freundenthal, 1940:17). In a mathematical context, where "bad induction" is supposed to be excluded (as Freundenthal remarked – 1940:37), no adjective is necessary to qualify the term *induction*.

2 "Die Mengentheoretiker sind gewöhnlich der Ansicht, dass der Begriff der ganzen Zahl definiert werden soll, und die vollständige Induktion bewiesen werden soll. Es ist aber klar, dass man nicht ins Unendliche definieren oder begründen kann; früher oder später kommt man zu dem nicht weiter Definierbaren bzw. Beweisbaren. Es ist dann nur darum zu tun, dass die ersten Anfangsgründe etwas unmittelbar Klares, Natürliches und Unzweifelhaftes sind. Diese Bedingung ist für den Begriff der ganzen Zahl und die Induktionsschlüsse erfüllt, aber entschieden nicht erfüllt für mengentheoretische Axiome der Zermelo'schen Art oder ähnliches; sollte man die Zurückführung der ersteren Begriffe auf die letzteren anerkennen, so müssten die mengentheoretischen Begriffe einfacher sein und das Denken mit ihnen unzweifelhafter als die vollständige Induktion, aber das läuft dem wirklichen Sachverhalt gänzlich zuwider."

night succeeds the day, but neither the day nor the night *causes* night or day. We shall argue below that whenever a multiplicity is grasped *collectively* (i.e., *at once*), an appeal is made to our *intuition of space*. Weyl says that the starting-point of mathematics is the series of natural numbers, the law according to which the number 1 is brought forth from nothing and where every number in turn gives rise to its successor (1921:57). On the next page Weyls calls the "always another one" the original mathematical intuition (*die mathematische Urintuition*) – it is not possible and it is not required to provide a further foundation for this *Urintution*. Upon this basis Weyl categorically holds that from the intuitionistic stand-point *complete induction* secures mathematics from being an enormous tautology and impregnates its statements as synthetic (non-analytic).[1]

With a reference to Weyl also Skolem mentions the fact that the concept of an integer and of induction constitutes the logical content of Hilbert's metamathematics.[2] Number displays an *order of succession* which is embodied in the application of induction. It is therefore significant that already in 1838 De Morgan used the equivalent expression "*successive induction*" (Freundenthal, 1940:36-37). Weyl does not stop to emphasize that mathematics in its entirety, even regarding the logical form in which it operates, is dependent upon the essence of the *natural numbers*.[3]

3.1 *Ordinality versus cardinality*

Those who are convinced that (axiomatic) set theory provides a sufficient basis for mathematics give priority to the concept of *cardinal number*. This concept seems to be more abstract and general than that of ordinal number, because in the latter case the relation between the members of a set is taken into account whereas it is absent in the case of *cardinal numbers*. According to Smart the main purpose of Cassirer's "critical study of the history of mathematics is to illustrate and confirm the special thesis that oridinal number is logically prior to cardinal number, and, more generally, that mathematics may be defined, in Leibnizian fashion, as the science of order" (1958:245).

The notion of a cardinal number faces two problems on the way to its foundational claim. First of all, it is dependent upon the concept of a *whole*, and

1 "Unabhängig aber davon, welchen Wert man dieser letzten Reduktion des mathematischen Denkens auf die Zweieinigkeit beimißt, erscheint vom intuitionistischen Standpunkt die vollständige Induktion als dasjenige, was die Mathematik davon bewahrt, eine ungeheure Tautologie zu sein, und prägt ihren Behauptungen einen synthetischen, nicht-analytischen Charakter auf" (1966:86).

2 "In der Tat basiert sich ja Hilbert sehr wesentlich auf dem Begriff der ganzen Zahl und der vollständigen Induktion in der Metamathematik, und diese stellt ja den logischen Inhalt seiner Theorie dar" (Skolem, 1929:89). What Skolem calls the "logical content" actually refers to the primtive meaning of *number* which is presupposed in logic – an insight also emphasized by Weyl (see the main text below and the next footnote)!

3 "die Mathematik ist ganz und gar, sogar den logischen Formen nach, in denen sie sich bewegt, abhängig vom Wesen der natürlichen Zahl" (1921:70).

secondly *cardinality* can only be established by implicitly applying *ordinality*. Surely the notion of *order* does not exhaust the meaning of number – as realized by Cassirer: "As soon as we proceed from the mere succession of numbers to a specific multiplicity (Vielheit) we encounter the transition from ordinal numbers to cardinal numbers as it was developed by Dedekind, Helmholtz and Kroneker" (1910:53).

The intimate link between the notion of a *cardinal number* and the supposition of a *totality* is also seen from the perspective of the concept of a one-to-one correspondence – Cassirer uses the term "Gleichzahligkeit" (1910:58) – which clearly presupposes the intuition of *at once* (simultaniety; Gleichzeitigkeit). The concept of *equivalence* implied in a one-one correlation underlies the difference between a straightforward *multiplicity* (how many/Wieviel) and the concept "just as many" / "equalling" (Gleich-viel) (Cassirer, 1910:62).

The crucial question is whether the intuition of a *whole* (*totality*), which is *given at once*, stems from our numerical intuition of succession or whether it has an irreducible meaning transcending the confines of relations of number? Is it possible to bring *succession* and *simultaneity* under the same denominator or are they irreducibly different? Skolem mentions the fact that Zermelo pointed out that the expressions *ordinal numbers* and *cardinal numbers* are employed when *sets* are compared (Skolem, 1929:80). But what is required to compare sets? How is it possible to establish the equality of two cardinal numbers?

The only way to achieve this is to order the members of both sets one by one, because otherwise no on-to-one correspondence could be assumed. In other words, *comparing* cardinals presupposes *some or other* order of succession! On the basis of a similar argument Weyl (also employing the term "Gleichzahligkeit") is therefore fully justified in his claim that "ordinal number is primary".

> It has been contested many times whether cardinal number is the primary concept and ordinal number the secondary. ... This definition is not only limited to finite sets; latching on to it Cantor developed his theory of infinite cardinal numbers within the context of his general set theory. But the possibility of mapping, present in the criterion of equi-numerosity, can only be assessed if the act of mapping takes the one after the other, in an ordered temporal succession and when the elements of both sets are thus themselves ordered. If the comparison of two sets is in an abstract way separated in determining the number of each set and subsequently comparing the numbers, then it is not allowed to order the individual sets themselves by designating one element after the other in time (as it is any way necessary for a set to be given individually; and numbers concern only sets such as these, that we use in everyday life). Therefore it seems to me incontestable that the *ordinal number is primary*. Modern

mathematical foundaitonal research, once again disrupted by dogmatic set theory, confirms this throughout.[1]

Dummett considers it to be a shortcoming that Frege did not pay enough attention to Cantor's work. Had he done that he "would have understood what it revealed, that the notion of an ordinal number is more fundamental than that of a cardinal number" (1995:293). Dummett continues by pointing out that this is even true of the finite case: "after all, when we count the strokes of a clock, we are assigning an ordinal number rather than a cardinal. If Frege had understood this, he would therefore have characterised the natural numbers as finite ordinals rather than as finite cardnals" (Dummett, 1995:293).

Cassirer is correct when he emphasizes that the "determination of number by the equivalence of classes presupposes that these classes themselves are given as a plurality" (1923:52). Although he holds that the transition from *ordinal number* to *cardinal number* does not entail any new mathematical content, he does discern a "new logical function" in the "formation of the cardinal number" (1953:42): "As in the theory of ordinal number the individual steps are established and developed in definite sequence, so here the necessity is felt of comprehending the series, not only in its successive elements, but as an ideal *whole*" (Cassirer, 1953:42). Cassirer does not realize that the supposed "new logical function" actually unmasks a new key element transcending the mere awareness of a succession, namely the idea of a *whole*.

4 Wholeness and Totality – the irreducibility of the Whole-Parts Relation

Although the terms "whole" and totality" are closely related to the term "continuity" it seems difficult to *define* the meaning of *continuous extension* – also realized by Dantzig: "From time immemorial the term *continuous* has been applied to space, ..., something that is of the same nature in its smallest parts as it is in its entirety, something *singly connected*, in short *something continuous*! don't you know .... any attempt to formulate it in a precise definition invariably ends in an impatient: 'Well, you know what I mean!'" (Dantzig,

1 Es ist viel darüber gestritten worden, ob nicht umgekehrt die Kardinalzahl das Erste und die Ordinalzahl der sekundäre Begriff sei. … Es ist diese Definition nicht einmal auf endliche Mengen beschränkt; die an sie sich knüpfende Theorie der unendlichen Kardinalzahlen hat G. *Cantor* im Rahmen seiner allgemeinen Mengenlehre entwickelt. Aber die Möglichkeit der Paarung, von der im Kriterium der Gleichzahligkeit die Rede ist, läßt sich nur prüfen, wenn die Zuordnungsakte einer nach dem andern, in geordneter zeitlicher Folge, vorgenommen und damit die Elemente beider Mengen selber geordnet werden. Reißt man die Vergleichung zweier Mengen abstraktiv auseinander in Zahlbestimmung jeder Menge und nachfolgenden Vergleich der Zahlen, so ist es also unerläßlich, die einzelne Menge selber zu ordnen durch Aufweisung eines Elementes nach dem andern in der Zeit (wie es ohnehin nötig ist, damit ein Inbegriff individuell gegeben sei; und nur von solchen Inbegriffen handeln die Zahlen, deren wir uns im täglichen Leben bedienen). Daher scheint es mir unbestreitbar, daß die *Ordinalzahl das Primäre* ist. Die moderne mathematische Grundlagenforschung, welche die dogmatische Mengenlehre wieder zerstört hat, bestätigt dies durchaus (Weyl, 1966:52-53).

1947:167).). Synonyms like "uninterrupted," "connected," "coherent," and so on, simply repeat what is meant by continuity, in stead of *defining* it! Yet, that a "continuous" whole allows for an *infinite number of divisions* was already discovered by the Greeks. Zeno's B Fr. 3 reads:

> If things are a multiplicity, then it is necessary that their number must be identical to their actual multiplicity, neither more nor less. But if there are just as many as there are, then their number must be limited (finite). If things are a multiplicity, then necessarily they are infinite in number; for in that case between any two individual things there will always be other things and so on. Therefore, then, their number is infinite.

The assumption that *things are many* serves two opposite conclusions! Apparently the two *sides* of the (spatial) whole-parts relation lies at the foundation of this argument. If the multiplicity of the first section refers to the many *parts* of the world as a *whole*, it stands to reason that taken together they constitute the *unity* of the world as a *whole* (and that their number would be limited). If, on the other hand, one starts with the *whole* and then tries to account for its *parts*, one must keep in mind that between any two of the many parts there will always be other, indicating an *infinity* of them. H. Fränkel explicitly uses the *whole-parts relation* to explain the meaning of this fragment (1968:425 ff., 430).

If this interpretation is sound, then Zeno not only understood something of the whole-parts relation, but also, for the first time, realized that spatial continuity is characterized by being *infinitely divisible*. This perspective can also be used in support of the interpretation given to Zeno's B Fr.1 by Hasse and Scholz (1928:10-13). This first fragment (which we inherited from Simplicius) states that if there exists a multiplicity, then simultaneously it must be large and small; large up to infinity and small up to nothingness. Hasse and Scholz clarify this fragment by interpreting it as follows:

> If it is permissible to conceptualize a line-stretch as an aggregate of infinitely many small line stretches, then there are two and only two possibilities. Every basic line segment either has a finite size (larger than zero), in which case the aggregate of line-stretches transcends every finite line-stretch; or the supposed line-stretches are zero-stretches in the strict sense of the word, in which case the composed line is also a zero-stretch, because the combination of zero-stretches can always only produce a zero-stretch, however large the number of zero-stretches used may be (1928:11).[1]

Besides the fact that we can render the two mentioned fragments of Zeno perfectly intelligible by using the whole-parts relation, further support for this understanding may also be drawn from the account which Aristotle gave of Zeno's arguments (cf. *Metaph*. 233 a 13 ff and 239 b 5 ff.). One of the standard expositions of Zeno's argumentation against the reality of motion is completely dependent on the employment of the whole-parts relation with its im-

1 We shall later on see that on the basis of measure theory and non-denumerable sets Grünbaum tried to side-step the last remark in order to accomplish an arithmetization of the continuum.

plied trait of infinite divisibility. Guthrie explains this argument by saying that according to Zeno

> Motion is impossible because an object moving between any two points A and B must always cover half the distance before it gets to the end. But before covering half the distance it must cover half of the half, and so ad infinitum. Thus to traverse any distance at all it must cover an infinite number of points, which is impossible in any finite time (1980:91-92).

For the larger part of 2000 years continuity (with its implied whole-parts relation), dominated the scene – both within the domains of philosophy and mathematics. Early modernity does witness atomistic theories of nature, but only in the course of the nineteenth century it penetrated the mathematical treatment of continuity. Bolzano explored an atomistic approach and it was brought to fruition by Weierstrass, Dedekind and Cantor.

The new assumption in the approach of Weierstrass holds that we have to define *limits* in terms of a *static domain* encompassing *all* real numbers. Boyer (1959:286) refers to this in the following explanation:

> In making the basis of the calculus more rigorously formal, Weierstrass also attacked the appeal to the intuition of continuous motion which is implied in Cauchy's expression – that a variable approaches a limit. Previous writers generally had defined a variable as a quantity or magnitude which is not constant; but since the time of Weierstrass it has been recognized that the ideas of variable and limit are not essentially phoronomic, but involve purely static considerations. Weierstrass interpreted a variable x simply as a letter designating any one of a collection of numerical values. A continuous variable was likewise defined in terms of static considerations: If for any value x_o of the set and for any sequence of positive numbers $d_1, d_2, \ldots\ldots, d_n$, however small, there are in the intervals $x_o - d_i, x_o + d_i$ others of the set, this is called continuous.

Weyl characterizes the new apparent success of the aim to arithmatize mathematics as *atomistic* (1921:56, 72). According to him within a continuum it is certainly possible, through divisions, to generate partial *continua*, but it is not clever (“unvernünftig”) to assert that the total continuum is composed out of the limits and these partial *continua*. Two years later, in 1923, he remarks that certain derived concepts come closer to an indication of the domain of study of geometry, such as a striaght line, and the occurrence of point and line (see Lorenzen, 1986:151). In general Weyl holds that “[A] true continuum after all coheres within itself and cannot be divided into separate pieces; it contradicts its essence.”[1] Weyl mentions that he changed his own view by accepting the position taken by Brouwer in this regard. Nonetheless one does not have to

1 “Erscheint dies dem heutigen Mathematiker mit seiner atomistischen Denkgewöhnung anstößig, so war es in früheren Zeiten eine allen selbstverständliche Ansicht: innerhalb eines Kontinuums lassen sich wohl durch Grenzsetzung Teilkontinuen erzeugen; es ist aber unvernünftig, zu behaupten, daß das totale Kontinuum aus der Grenze und jenen Teilkontinuen zusammengesetzt sei. Ein wahrhaftes Kontinuum ist eben ein in sich Zusammenhängendes und kann nicht in getrennte Bruchstücke aufgeteilt werden; das widerstreitet seinem Wesen” (Weyl, 1921:73).

adhere to the intuitionistic understanding of (infinity and) continuity to realize that the whole-parts relation and the totality-character of continuity stands in the way of a complete (atomistic) arithmetization of continuity. Paul Bernays, the co-worker of David Hilbert, senses the irreducibility of the spatial whole-parts relation (i.e., the *totality* feature of spatial continuity) with an astonishing lucidity:

The property of being a totality "undeniably belongs to the geometric idea of the continuum. And it is this characteristic which obstructs a complete arithmetization of the continuum."[1]

Compare his remark in a different context where he even states that the classical foundation of the real numbers given by Cantor and Dedekind does not "manifest a complete arithmetization" (1976: 187-188). To this he adds the remark:

> It is in any case doubtful whether a complete arithmetization of the idea of the continuum could be justified. The idea of the continuum is any way originally a geometrical idea (1976:188).

Something remarkable emerges from this situation. If the nature of totality (wholeness / the whole-parts relation) cannot be arithmetized how does one explain the entailed property of being infinitely divisible? Does this property imply that the (numerical) intuition of succession (literally without an end / endless / infinite) is an indispensible building block of continuity? Does it therefore mean that "number" is "built into" the nature of (spatial) "continuity"?

5 Some crucial structural features

At the beginning of our reflections we have mentioned the opposing one-sidedness present in the history of mathematics and philosophy regarding the relationship of discreteness and continuity. We have highlighted the key role of terms such as *order*, *succession*, *induction*, *ordinal number*, *cardinal number*, *infinity* (endlessness), *at once*, *wholeness* (totality), the *whole-parts relation* and (infinite) *divisibility*. In addition we have mentioned the basic role of *multiplicity* which, within logic (and mathematics), is reflected in the concepts of (*constants* and) *variables*. On top of that we had to refer to the *connectives* of first order predicate calculus [negation, conjunction, disjunction and (bi-)conditional].

Sometimes disciplines do pay attention to the supposedly *basic concepts* they employ. We have already suggested that these *basic concepts* are closely related to the primitive terms encountered in a discipline and the implied relation between *unity* and *multiplicity* is centered in the way in which one accounts for the *coherence of irreducibiles*. Rucker, for example, elsewhere remarks: "The discrete and continuous represent fundamentally different aspects of the mathematical universe" (Rucker, 1982:243). Moore proceeds a step further by discerning two 'clusters' of concepts which dominate the his-

1 "Und es ist auch dieser Charakter, der einer vollkommenen Arithmetisierung des Kontinuums entgegensteht" (1976:74).

tory of the notion of infinity. In the first cluster the following terms are categorized: “boundlessness; endlessness; unlimitedness; immeasurability; eternity; that which is such that, given any determinate part of it, there is always more to come; that which is greater than any assignable quantity” (1990:1). Within the second cluster he mentions: “completeness; wholeness; unity; universality; absoluteness; perfection; self-sufficiency; autonomy” (Moore, 1990:1-2).

These characterizations are congruent with the quoted weak spot alluded to by Fraenkel *et. al* – regarding the , “the gap between discrete and continuous” (1973:212-213). On the previous page they present a stronger formulation: “*Bridging the gap between the domains of discreteness and of continuity*, or between arithmetic and geometry, is a central, presumably even *the* central problem of the foundation of mathematics” (Fraenkel, A., et al., 1973:211). Though this formulation may be consistent with the history of the problem (which explored the one-sided options of reducing number to space or reducing space to number), it may be fruitful alternatively to acknowledge the *difference* between discreteness and continuity while holding on to the *mutual connectedness* of both these perspectives.

However, exploring this alternative does require distinctions not familiar within the special sciences. The guiding perspective required is an account of the *analogical basic concepts* employed by the disciplines. Though not universally acknowledged by the various disciplines the reality of using such concepts is ubiquitous.

6 The inevitability of employing analogical elementary basic concepts

Contemporary postmodernity emphasizes *language* as horizon, but in doing that it does not sufficiently distinguish between (i) *entitary analogies* and (ii) *modal analogies*. Ordinary and scientific language designate analogies between entities with *metaphors* (such as the *foot* of the *mountain* or the *wave* theory of light). Whereas such metaphors (*entitary analogies*) could be replaced by entirely *different* ones, the substitution of *modal analogies* is only meaningful if *synonomous terms* are employed. A spatial term such as *domain*, for example, could be replaced by others, such as *range, scope* or *sphere*. Although, for technical purposes, mathematicians may decide to attach slightly different connotations to these terms, the fact that they all share a *generic spatial meaning* cannot be denied. In order to understand what is at stake in the case of truly *analogical basic concepts* we look at the classical legacy about the *infinite divisibility* of continuity. In his article on the infinite commemorating Karl Weierstrass (published in 1925), Hilbert looks at this issue from the perspective of the *infinitely small* and the *infinitely large* (1925:163 ff.). The discovery of *energy quanta* on the one hand and Einstein's *theory of relativity*, on the other hand, eliminate both possibilities.[1] The following distinction is therefore necessary: that between *mathematical space* and *physical space*. Whereas the former – in a purely *abstract* and *functional* perspective – is both *continuous* and *infinitely divisible*, *physical space* is neither *continuous* nor *infinitely divisible*. Since it is bound to the *quantum structure* of energy physical space cannot be subdivided *ad infinitum*. Energy quanta indeed represent the *limit* of the divisibility of energy.[2]

An analogy is present whenever differences are shown in what is similar. In this case: both mathematical space and physical space are extended (their *similarity*), but in being discontinuous and not infinitely divisible (their differences) the latter differs from the former.

The concept physical space is an instance of an analogical basic concept. It is based upon the fact that within the structure of the physical aspect the meaning of space is analogically reflected. The multiplicity of properties combined in the unity of a (logical) concept, in turn, analogically reflects the meaning of number (one and many) within the analytical aspect of reality. In general it is the case that each aspect of reality expresses its coherence with the other aspects through analogical moments.

The Dutch philosopher, Herman Dooyeweerd, created a unique theory of "modal law-spheres" to explain this complex state of affairs. The classical distinction between *properties* and *entities* runs parallel to that between *function* and *entity*.[3] Entities *function* in the various aspects of reality.That is to say, the various aspects have an *ontic* status and are not mere *products of thought*. Since Descartes an implicit *nominalist assumption* accompanies the understanding of functional (modal) aspects.

Remark: The classical realist position in philosophy accepts a threefold existence of the so-called "universalia": *ante rem* in God's mind (the legacy of Plato's transcendent ideal forms), *in re* within things (*a la* Aristotle – as their universal substantial forms), and *post rem* (as subjective universal concepts – with as criterion of truth the *correspondence* between thought and reality ("adequatio intellectus et rei"). Nominalism rejects the first two and only acknowledges universal concepts (or: words) within the human mind – outside the human mind only

1 A more extensive explanation is found in his oration on *Naturerkennen und Logik* (Hilbert, 1930:380-381). In order to account for the *discrete* nature of the omission or absorbtion of energy, Planck postulated that radiant energy is *quantized*, proportional to the frequency v in the formula $E = hnv$ – where n is an integer, v the frequency, and h the quantum of action (*Wirkungsquantum*) with the value 6.624×10^{-34}.

2 Owing to gravitation ("curved space") the universe is considered to be *finite* though *unbounded*.

3 The quoted work from Cassirer, *Substanzbegriff und Funktionsbegriff*, made an extremely valuable contribution to an understanding of this issue. Often function concepts are also designated as concepts of relation ("Relationsbegriffe").

> uniquely individual entities exist. (Within nominalism a distinction is made between those nominalists who accept universal concepts: conceptualism and the more extreme position which only accepts words.) In their ordinary understanding *sets* are *universals* and therefore they partake, in the words of Fraenkel et al., in "the well-known and amply discussed classical problem of the ontological status of the universals" (1973:332). The three main traditional answers given to this problem, namely realism, nominalism and conceptualism, are related with their modern counter-parts known as platonism, neo-nominalism, and neo-conceptualism (Fraenkel et al., 1973:332). Stegmüller explains the mathematical relevance of these different positions as follows: the three mentioned ontological positions, namely nominalism, conceptualism, and platonism, are mapped in terms of the quantitative categories "finite totality (Gesamtheit) – denumerable infinite totality – non-denumerable infinite totality" (1965:117-118).

In his *Principles of Philosophy* Descartes says "that number and all universals are only modes of thought" (Part I, LVIII). David Hilbert (implicitly) still continues this orientation when he argues that after we have established the finiteness of reality in two directions (the infinitely small and the infinitely large), it may still be the case that the infinite does have a justified place *within our thinking*![1] The implication is clear: "reality" is exhausted by *entities*, modal (functional) properties are located in the *realm of thought*.

7 The theory of modal aspects

The theory of *modal law-spheres* first of all acknowledges the *ontic givenness* of the modal aspects. Hao Wang remarks that Gödel is very "fond of an observation that he attributes to Bernays": "That the flower has *five* petals is as much part of objective reality as that its color is *red*" (Wang, 1982:202). The quantitative side (aspect) of things (entities) is not a *product* of thought – at most human reflection can *explore* this given (functional) trait of reality by analyzing what is entailed in the *meaning* of multiplicity. Yet, in doing this (theoretical and non-theoretical) thought explores the *given* meaning of this quatitative aspect in various ways, normally first of all by *forming* (normally called: *creating*) **numerals** (i.e., number symbols). The simplest act of counting already explores the *ordinal meaning* of the quantitative aspect of reality. Frege correctly remarks "that counting itself rests on a one-one correlation, namely between the number-words from 1 to *n* and the objects of the set" (quoted by Dummett, 1995:144).

However, in the absence of a sound and thought-through distinction between the *dimension of concretely existing entities* (normally largely identi-

1 "Die Endlichkeit des Wirklichen haben wir nun in zwei Richtungen festgestellt: nach dem Unendlichkleinen und dem Unendlichgroßen. Dennoch könnte es sehr wohl zutreffen, daß das Unendliche *in unserem Denken* einen wohlberechtigten Platz hat und die Rolle eines unentbehrlichen Begriffes einnimmt" (1925:165).

fied with "physical" or "space-time existence") and the *dimension of functional modes* (aspects) of ontical reality, which cannot be observed through sensory perception, mathematicians oftentimes struggle to account for the epistemic status of their "subject matter." Perhaps the awareness for the need of acknowledging this distinct dimension of reality is best articulated in Wang's discussion of Gödel's thought. Wang discusses Gödel's ideas regarding "mathematical objects" and mentions his rejection of Kant's conception that they are "subjective." Gödel holds: "Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality" (quoted by Wang, 1988:304, cf. p.205). To this Wang adds his support: "I am inclined to agree with Gödel, but do not know how to elaborate his assertions. I used to have trouble by the association of objective existence with having a fixed 'residence' in spacetime. But I now feel that 'an aspect of objective reality' can exist (and be 'perceived by semiperceptions') without its occupying a location in spacetime in the way physical objects do" (Wang, 1988:304).

Of course Wang could have referred to the important insights of Cassirer in this regard. Already in his article on Kant and modern mathematics (1907), and particularly in his influential work: *Substance and Fucntion* (1910), Cassirer distinguishes between *entities* and *functions*. He crearly realizes that quantitative properties are not exhausted by any individual entity: ""number is to be called universal not because it is contained as a fixed property in every individual, but because it represents a constant condition of judgment concerning every individual as an individual" (1953:34). If we set aside the (neo-)Kantian undertones of this statement, Cassirer already saw something of the *modal universality* of the arithmetical aspect of reality.

Every aspect has an undefinable core (or: nuclear) meaning (also designated as the meaning-nucleus) which *qualifies* all the analogical meaning-moments within a specific aspect. These analogical moments may refer backwards to *ontically earlier* aspects (known as *retrocipations*) or forwards to *ontically later* aspects (known as *anticipations*). *Earlier* and *later* are taken in the sense of *cosmic time* as it is called by Dooyeweerd. The aspects of reality are fitted in an inter-modal coherence of earlier and later: the most basic aspect is that of number (meaning-nucleus: discrete quantity), which is followed by the aspect of space (continuous extension), the kinematical aspect (core: constancy), the physical (change/energy-operation), the biotic (life), the sensitive (feeling), the logical (analysis), the cultural-historical (formative control/power), the sign-mode (symbolical signification), and so on. Cosmic time has an order-side and a duration-side – we are acquainted with the numerical time-order of succession, the spatial time-order of simultaneity, the kinematical time-order of uniform flow, the irreversible physical time-order, the biotical time-order of birth, growth, maturation, aging and dying (cf. Dooyeweerd, 1996-I, pp.30-32).

The meaning of an aspect finds expression in its coherence with other aspects (retrocipations and anticipations). *Retrocipatory analogies* are captured in the *elementary basic concepts* of a discipline.

The first challenge in an analysis of the *elementary (analogical) basic concepts* of the various academic disciplines is to identify the modal "home" or "seat" of particular terms.

Within an aspect their is a difference between the *order-side* (also known as the law-side) and its correlate, the *factual side* (that which is subjected to the law-side and *delimited* and *determined* by the latter). The numerical time-order of succession belongs to the law-side of the arithmetical aspect, and any *ordered* sequence of numbers appears at its factual side (think of the natural numbers in their normal succession). With the exception of the numerical aspect (which only have subject-subject relations), all the other aspects in addition also have subject-object relations at their factual side.[1] (Cf. Sketch 1on page 37).

7.1 *Numerical and spatial terms*

It is intuitively clear that our awareness of *succession* and *multiplicity* (underlying the concept of an ordinal number and induction) makes an appeal to the quantitative aspect of reality. These terms therefore have their modal "seat" ("home") in the arithmetical aspect.

Of course it is natural that special scientists will attempt to reduce apparently primitive terms to familiar and more basic ones. But if such an attempt becomes circular, or even worse, contradictory, then it may be the case that the primitive terms involved are truly *irreducible*! Phrased differently: an attempt to define what is undefinable may end up in *antinomic reduction*.[2] Sometimes the challenge is not to get *out* of the circle, but to get *into* it(s irreducible meaning)!

In the course of our preceding discussion the following cluster of terms probably transcend the confines of the numerical aspect: *simultaneity* (at once), *completedness*, *wholeness* (totality), and the *whole-parts relation*.

The most prominent recognition of the spatial "home" of wholeness and totality is found in the thought of Bernays. He writes that it is recommendable not to distinguish the arithmetical and geometrical intuition according to the moments of the spatial and the temporal, but rather by focusing on the difference between the *discrete* and the *continuous*.[3] Being fully aware of the

1 The identifiability and distinguishability of something represents its *latent* logical object-function. When it is identified and distinguished by a thinking subject, this analytical object-function is made *patent*.

2 The classical example is Zeno's attempt to define movement in static spatial terms – and just doing that eliminated the meaning of motion.

3 "Es empfiehlt sich, die Unterscheidung von "arithmetischer" und "geometrischer" Anschauung nicht nach den Momenten des Räumlichen und Zeitlichen, sondern im Hinblick auf den Unterschied des Diskreten und Kontinuierlichen vorzunehmen" (1976:81).

arithmeticistic claims of modern analysis it is all the more significant that Bernays questions the attainability of this ideal of a *complete arithmetization* of mathematics. He unambiguously writes:

> We have to concede that the classical foundation of the theory of real numbers by Cantor and Dedekind does not constitute a *complete* arithmetization of mathematics. It is anyway very doubtful whether a complete arithmetization of the idea of the continuum could be fully justified. The idea of the continuum is after all originally a geometric idea (Bernays, 1976:187-188).[1]

Particularly in explaining the difference between the potential and the actual infinite the difference between *succession* and *at once* and the irreducibility of the notion of a *totality* surfaces. Hilbert introduces the difference between the potential and the actual (or: genuinely) infinite by using the example of the "totality of the numbers 1, 2, 3, 4, ..." which is viewed as a unity which is given at once (completed):

> If one wants to provide a brief characterization of the new conception of infinity introduced by Cantor. one can indeed say: in analysis where the infinitely small and the infinitely large feature as limit concept, as something becoming, originating and generated, that is, as it is stated, with the potential infinite. But this is not the true infinite. We have the latter when, for example, we view the totality of the numbers 1, 2, 3, 4, ... as a completed unity or when we observe the points of a line as a totality of things, given to us as completed. This kind of infinity is designated as the actual infinite.[2]

According to Lorenzen the understanding of real numbers with the aid of the actual infinite cannot camouflage its ties with space (geometry):

> "The overwhelming appearance of the actual infinite in modern mathematics is therefore only understandable if one includes geometry in one's treatment. ... The actual infinite contained in the modern concept of real numbers still reveals its descent (Herkunft) from geometry" (1968:97).

Lorenzen highlights the same assumption when he explains how real numbers are accounted for in terms of the actual infinite:

> One imagines much rather the real numbers as all at once actually present – even every real number is thus represented as an infinite decimal fraction, as if

1 "Zuzugeben ist, daß die klassische Begründung der Theorie der reellen Zahlen durch Cantor und Dedekind keine *restlose* Arithmetisierung bildet. Jedoch, es ist sehr zweifelhaft, ob eine restlose Arithmetisierung der Idee des Kontinuums voll gerecht werden kann. Die Idee des Kontinuums ist, jedenfalls ursprünglich, eine geometrische Idee."

2 "Will man in Kürze die neue Auffassung des Unendlichen, der Cantor Eingang verschafft hat, charakterisieren, so könnte man wohl sagen: in der Analysis haben wir es nur mit dem Unendlichkleinen und dem Unendlichengroßen als Limesbegriff, als etwas Werdendem, Entstehendem, Erzeugtem, d.h., wie man sagt, mit dem potentiellen Unendlichen zu tun. Aber das eigentlich Unendliche selbst ist dies nicht. Dieses haben wir z. B., wenn wir die Gesamtheit der Zahlen 1, 2, 3, 4, ... selbst als eine fertige Einheit betrachten oder die Punkte einer Strecke als eine Gesamtheit von Dingen ansehen, die fertig vorliegt. Diese Art des Unendlichen wird als aktual unendlich bezeichnet" (Hilbert, 1925:167).

> the infinitely many figures (Ziffern) existed all at once (alle auf einmal existierten) (1972:163).

These modes of speech highlight the inevitability of employing terms with a *spatial descent* even when the pretention is to proceed purely in *numerical* terms. Lorenzen correctly points out that arithmetic by itself does not provide any motive for the introduction of the actual infinite (1972:159). The fundamental difference between arithmetic and analysis in its classical form, according to Körner, rests on the fact that the central concept of analysis, namely that of a real number, is defined with the aid of actual infinite totalities ("aktual unendlicher Gesamtheiten" – 1972:134). Without this supposition Cantor's proof on the non-denumerability of the real numbers collapses into denumerability. While rejecting the actual infinite, intuitionism interprets Cantor's diagonal proof of the non-denumerability of the real numbers in a constructive sense – cf. Heyting (1971:40), Fraenkel *et al.* (1973:256,272) and Fraenkel (1928:239 note 1). However, in order to reach the conclusion of non-denumerability, every constructive interpretation falls short – simply because there does not exist a *constructive* transition from the potential to the actual infinite (cf. Wolff, 1971).

Remark: Husserl's *Philosophie der Aritmetik*[1]

> In 1887 Edmund Husserl completed his habilitation with a study concerned with a psychological analysis of the concept of number – followed in 1891 by his extensive "Philosophie der Arithmetik". In this work he connected the infinite both with the psychological nature of a collective synthesis (1970:64 ff.), and with the principle of succession (p.220). He questioned the actual infinite in the light of the finiteness of arithmetic. He even planned a second volume of his "Philosophie der Arithmetik," but eventually realized that he could not achieve his aim without using the actual infinite. L. Eley (in the Preface to Husserliana, Vol. XII) saw in this failure the reason why this second volume was never published. From the unpublished manuscripts it is nevertheless clear that Husserl did not succeed in giving a foundation to general arithmetic without the acceptance of the actual infinite (which shows that he confronted himself with classical analysis).

It seems to be impossible to develop set theory without "borrowing" key-elements from our basic intuition of space, in particular the (order of) *at once* and its factual correlate: *wholesness* / *totality*. Since spatial subjects are *extended* their multiple parts exist all at once. This multiplicity is at the factual side of the spatial aspect a *retrocipation* to the meaning of number – i.e., *multiple parts* analogically reflect the meaning of number (multiplicity) within space.

Bernays did not have a theory of modal aspects at his disposal and therefore lacks the possibility of articulating explicitly the intermodal connections between number and space. Yet, in stead of saying that the mathematical analy-

1 Edmund Husserl completed his habilitation in 1887 and published in 1891 his study: *Philosophie der Aritmetik*.

sis of the *meaning of number* reveals an anticipation to the meaning of space, he states that the idea of the continuum is a geometrical idea which analysis expresses with an arithmetical language.[1]

8 Deepening our understanding of infinity

We have already pointed out that the most primitive meaning of infinity relates to the *arithmetical time-order of succession*. It has its foundation in the order-side or the law-side of the numerical aspect. This order of succession determines every infinitely proceeding sequence of numbers as well as the different operations at the law-side of the numerical aspect [operations such as *addition*, *multiplication*, *subtraction*, *division*, as well as the principle of induction – which can be disclosed by (non-theoretical as well as theoretical mathematical) thinking]. Because the expression *potential infinite* lacks an immediate intuitive clarity, the most appropriate phrase suitable to capture the meaning of this most basic kind of infinity is to call it the *successive infinite*.

We have noted that the Greeks already turned the infinite "inwards" – with their discovery of the *infinite divisibility* of continuity. In terms of the intermodal coherence between the aspects of number and space this infinite divisibility *analogically* (i.e., as a retrocipatory analogy) *reflects* the *successive infinite* at the law-side of the numerical aspect. The system of rational numbers (fractions), in turn, analogically (i.e., as an anticipatory analogy) reflects this infinite divisibility evinced at the factual side of the spatial aspect. Consequently, the rational numbers represent an *anticipation* to a *retrocipation* (see Sketch 2 on page 37).[2]

When, under the guidance of our theoretical (i.e., modally abstracting) insight into the meaning of the spatial order of simultaneity, the original modal meaning of the numerical time-order is disclosed (deepened), we encounter the *regulatively deepened anticipatory* idea of actual or completed infinity. Any sequence of numbers may then, directed in an anticipatory way by the spatial order of simultaneity, be considered *as if* its infinite number of elements are present as a *whole* (*totality*) *all at once*.

In this context it is noteworthy that Hao Wang informs us that Kurt Gödel speaks of sets as being "quasi-spatial" and then adds that he is not sure whether Gödel would have said the "same thing of numbers" (1988:202). This mode of speech is in line with our suggestion that the undefined term "element of" employd in ZF set theory actually harbours the *totality* feature of continu-

1 "Die Idee des Kontinuums ist, jedenfalls ursprünglich, eine geometrische Idee, welche durch die Analysis in arithmetischer Sprache ausgedrükt wird" (1976:74).

2 In passing we note that the numerical anticipation to a spatial retrocipation – revealed in the denseness of the rational numbers – is not sufficient to characterize any real number in a mathematical satisfactory way – at least not in a deepened and disclosed way. Yet intuitionism uses this *semi-disclosed* meaning of number in its characterization of the "continuum."

ity, because then it does entail that set theory (in an anticipating way) is dependent on “something spatial”!

This also amounts to a confirmation of the unbreakable coherence between the law-side and the factual side of the numerical and the spatial aspects. The modal anticipation from the numerical time-order to the spatial time-order must therefore have its correlate at the factual side. At the factual side of the numerical aspect we encounter the sequence of natural numbers and integers (expressing the primitive meaning of numerical discreteness). Introducing the *dense* set of rational numbers manifests (as an anticipation to a retrocipation) the *semi-disclosed* meaning of number.

When we employ the anticipation at the law-side of the numerical aspect to the law-side of the spatial aspect we encounter the intermodal foundation of the notion of *actual infinity* – although the basic intuitions at play suggest a more suitable designation for the actual infinite. But before we discuss a more appropriate naming of the actual infinite a brief remark is required to explain the “as if” character of this deepened notion of infinity. The anticipation from number to space at the law-side *determines* the correlated multiplicity of natural numbers, integers and rational numbers which are, under the employment of the actual infinite, considered *as if* they are present as completed (though infinite) *wholes* or *totalities*.

Remark: “As if”: the actual infinite as a regulative hypothesis

Vaihinger developed a whole philosophy of the “as if” (*Die Philosophie des Als Ob*), in which he tries to demonstrate that various special sciences may use, with a positive effect, certain *fictions* which in themselves are considered to be *internally antinomic*. The infinite, both in the sense of being infinitely large and infinitely small, is evaluated by Vaihinger as examples of necessary and fruitful fictions (cf. 1922:87 ff., p.530). Ludwig Fischer presents a more elaborate mathematical explanation of this notion of a *fiction*. In general he argues: “The definition of an irrational number by means of a formation rule always involves an 'endless', i.e. unfinished process. Supposing that the number is thus given, then one has to think of it as the completion (Vollendung) of this unfinished process. Only in this ... the internally antinomic (in sich widerspruchsvolle) and *fictitious* character of those numbers are already founded” (1933:113-114). Without the aid of a preceding analysis of the modal meaning of number and space, this conclusion is almost inevitable. Vaihinger and especially Fischer simply use the number concept of *uncompleted infinity* (the successive infinite) as a standard to judge the (onto-)logical status of the actual infinite. Surely, within the closed (not yet deepened) meaning of the numerical aspect, merely determined by the arithmetical time-order of *unfinished succession*, the notion of an actual infinite multiplicity indeed is *self-contradictory*.

However, the meaning intended by us for the actual infinite *transcends* the limits of this concept of number since, in a regulative way, it refers to the core meaning of the spatial aspect which (in an anticipatory sense) underlies the *hy-*

pothetical use of the time-order of *simultaneity* (the "all" viewed as being present *at once*).

Paul Lorenzen echoes something of this approach in his remark that the meaning of actual infinity as attached to the "all" shows the employment of a fiction – "the fiction, as if infinitely many numbers are given" (1952:593). In this case too, we see that the "as if" is ruled out, or at least disqualified as something *fictitious*, with an implicit appeal to the primitive meaning of number.

As long as one sticks to the notion of a *process*, one is implicitly applying the yardstick of the *successive infinite* to judge the actual infinite.

Paul Bernays did see the essentially *hypothetical* character of the *opened up* meaning of number, without (due to the absence of an articulated analysis of the modal meaning coherence between number and space) being able to exploit it fully: "The position at which we have arrived in connection with the theory of the infinite may be seen as a kind of the philosophy of the 'as if'. Nevertheless, it distinguishes itself from the thus named philosophy of Vaihinger fundamentally by emphasizing the consistency and trustworthiness of this formation of ideas, where Vaihinger considered the demand for consistency as a prejudice ..." (1976:60).

Although the deepened meaning of infinity is sometimes designated by the phrase *completed infinity*, this habit may be misleading. If *succession* and *simultaneity* are irreducible, then the idea of an infinite totality cannot simply be seen as the completion of an *infinite succession*. When Dummett refers to the classical treatment of infinite structures "as if they could be completed and then surveyed in their totality" he equates this "infinite totality" with "the entire output of an infinite process" (1978:56). The idea of an infinite totality transcends the concept of the successive infinite.

A remarkable ambivalence in this regard is found in the thought of Abaraham Robinson. His exploration of *infinitesimals* is based upon the meaning of the at once infinite. A number a is called *infinitesimal* (or *infinitely small*) if its absolute value is less than *m* for all positive numbers *m* in $\Re$ ($\Re$ being the set of real numbers). According to this definition 0 is *infinitesimal*. The fact that the infinitesimal is merely the correlate of Cantor's transfinite numbers is apparent in that *r* (*not equal to* 0) is infinitesimal if and only if *r to the power minus 1* (r^{-1}) is infinite (cf. Robinson, 1966:55ff). In 1964 he holds that "infinite totalities do not exist in any sense of the word (i.e., either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally, *meaningless.*" Yet he believes that mathematics should proceed as usual, "i.e., we should act *as if* infinite totalities really existed" (Robinson, 1979:507).

Cantor explicitly describes the *actual infinite* as a constant quantity, *firm and determined in all its parts* (1962:401). Throughout the history of Western philosophy and mathematics, all supporters of the idea of *actual infinity* implicitly or explicitly employed *some* form of the *spatial order of simultaneity*. What should have been used as an *anticipatory regulative hypothesis* (the idea of *actual infinity*), was often (since Augustine) reserved for God or an eternal being, accredited with the ability to oversee any infinite multiplicity *all at once*.

This anticipatory regulative hypothesis of actual infinity does not *cancel* the original modal meaning of number, but only *deepens* it under the guidance of theoretical thought.

Perhaps also here it would be helpful to introduce new terms for this well-known expression. In stead of speaking about the *actual infinite* we should rather talk about the *at once infinite*. Alongside the successive infinite the at once infinite already surfaced in the disputes of the early 14th century about the infinity of God.[1]

These new expressions relate directly to our basic numerical and spatial intuitions, viz., our awareness of *succession* and *simultaneity* – and their mutual irreducibility is based upon the irreducibility of the aspects of number and space.[2]

A truly deepened and disclosed account of the real numbers cannot be given without the aid of the *at once infinite*. That this *anticipatory coherence* between number and space always functioned prominently in a deepened account of the real numbers, may be shown from many sources. It will suffice to mention only *one* in this context. But before we do that we have to return briefly to the relationship between mathematics and logic.

9 Mathematics and Logic

If the meaning of analysis presupposes the original quantitative meaning of unity and multiplicity, turning the logicistic attempt to deduce the meaning of number from the logical mode into a vicious circle, then it appears to be strange that we still have to concede that the discipline of mathematics is in need of a logical foundation?!

Cassirer first of all approaches this problem in terms of the numerical analogy within the logical-analytical mode. His question is that it is not understandable why one only accepts logical identity and difference, which enter the set concept as necessary elements, as such basic functions, but that one does not do the same with regard to numerical unity and difference? He claims that a truly satisfactory deduction of the one from the other is also not

1 Compare the expressions *infinitum successivum* and *infinitum simultaneum* (Maier, 1964: 77-79).

2 Dooyeweerd did not accept the idea of the *at once infinite* (*actual infinity*) owing to the fact that he was strongly influenced by the intuitionistic mathematicians Brouwer and Weyl in this regard. Cf. Dooyeweerd, 1996-I:98-99 (footnote 1) and 1996-II:340 (footnote 1).

achieved by set theory, which entails a persistent suspicion that all similar attempts will continue to harbor a concealed epistemological circle.[1]

Yet, although it is true that the meaning of analysis presupposes the meaning of number, the (theoretical) analysis of number remains a thought act which is qualified by the logical aspect. This explains why the discipline of logic occupies such a founding role with respect to all disciplines, including mathematics.

10 The circularity entailed in set theoretical attempts to arithmetize continuity

The nuclear meaning of space is *indefinable*. If one tries to define the *indefinable* two equally objectionable options are open:

(i) either one ends up with a *tautology* – coherence, being connected, and so on, are all synonymous terms for *continuity* – or, even worse,

(ii) one becomes a victim of (antinomic) *reduction*, i.e. one tries to reduce what is indefinable to something familiar but distinct.

While the idea is ancient, modern Cantorian set theory again came up with the conviction that a spatial subject such as a particular line must simply be seen as an infinite (technically, a non-denumerable infinite) set of points.

If the points which constitute the one dimensional continuity of the line were themselves to possess any extension whatsoever, it would have had the absurd implication that the continuity of every point is again constituted by smaller points than the first type, although necessarily they also would have had some extension. This argument could be continued *ad infinitum*, implying that we would have to talk of points with an ever-diminishing "size." In reality such "diminishing" points do not at all refer to real points, since they simply reflect the nature of continuous extension, which as we have seen, is *infinitely divisible*. Such points build up space out of space.

Anything which has *factual extension* has a subject-function in the spatial aspect (such as a chair) or is a modal subject in space (such as a line, a surface, and so forth). A point in space, however, is always dependent on a spatial subject since it does not itself possess any extension.

Remark: We are now in a better position to explain the structural meaning of the three undefined terms introduced by Hilbert in his mentioned axiomatization of geometry in 1899. These terms instantiate the spatial subject-object relation at the factual side of the spatial aspect. The term "line" represents the factual subject-side (factual, one-di-

1 "In der Tat ist nicht einzusehen, warum man lediglich logische Identität und Verschiedenheit, die als notwendige Momente in den Mengenbegriff eingehen, als solche Urfunktionen gelten lassen und nicht auch die numerische einheit und den numersichen Unterschied von Anfang an in diesen Kreis aufnehmen will. Eine wirklich befriedigende Herleitung des einen aus dem anderen ist auch der mengentheoretischen Auffassung nicht gelungen, und der Verdacht eines verstckten erkenntnistheoretischen Zirkels blieb gegenüber allen Versuchen, die in dieser Richtung gemacht werden, immer bestehen" (Cassirer, 1957:73-74).

mensional spatial extension); the term "point" represents the factual object-side; while the relation of dependence of the latter upon the former is represented by the "relational term": "lies on".

The length, surface or volume of a point is always zero – it has none of these. If the measure of one point is zero, then any number of points would still have a zero-measure. Even a(n denumerable) infinite set of points would never constitute any positive distance, since distance presupposes an extended subject.[1]

Remark: The following classical "definition" of a line is well-known: *A straight line is the shortest distance between two points*. A straight line, however, is a factual spatial figure extended in one dimension. The measure of this extension is indicated by the numerical analogy of distance (magnitude). We can say in a particular instance that the length (i.e. the numerical analogy) of a line is so much. The so much of a line, however, is not the line. In other words, the extension of the line cannot be defined by the indication of its *length*. The length of a line presupposes the *factual extension* of the line – from which it remains *distinct*. For this reason Hilbert imported the term *line* as an *undefined* term in his mentioned axiomatic foundation of geometry (cf. 1899). The fourth problem mentioned by him in his presentation to the International Mathematical Conference at Paris in 1900 correctly refers to the *problem of the straight line* as the *shortest connection of* (and not: **distance between**!) *two points* (Hilbert, 1970:302).

A. Grünbaum has combined insights from the theory of point-sets (founded by Cantor) with general topological notions and with basic elements in modern dimension theory in order to arrive at an apparently consistent conception of the extended linear continuum as an aggregate of unextended elements (1952:288 ff.). From his analysis it is clear that he actually had "unextended unit point-sets" in mind and not simply a set of "unextended points" (1952: 295). Initially he starts with a non-metrical topological description and then, later on, introduces a suitable metric normally used for Euclidean spaces (point-sets). The all-important presupposition of this analysis is the acceptance of the linear Cantorean continuum (arranged in an order of magnitude, i.e. the class of all real numbers) (cf. 1952:296).

1 The following classical "definition" of a line is well-known: *A straight line is the shortest distance between two points*. A straight line, however, is a factual spatial figure extended in one dimension. The measure of this extension is indicated by the numerical analogy of distance (magnitude). We can say in a particular instance that the length (i.e. the numerical analogy) of a line is so much. The so much of a line, however, is not the line. In other words, the extension of the line cannot be defined by the indication of its *length*. The length of a line presupposes the *factual extension* of the line – from which it remains *distinct*. For this reason Hilbert imported the term *line* as an *undefined* term in his mentioned axiomatic foundation of geometry (cf. 1899). The fourth problem mentioned by him in his presentation to the International Mathematical Conference at Paris in 1900 correctly refers to the *problem of the straight line* as the *shortest connection of* (and not: **distance between**!) *two points* (Hilbert, 1970:302).

On the basis of certain distance axioms, the real function d(x,y) (called the distance of the points x,y which have the coordinates x_i,y_i) is used to define the length of a point-set constituting a finite interval on a straight line between two fixed points (the number of this distance is its *length*). For example, the length of a finite interval (a,b) is defined as the non-negative quantity b – a (disregarding the question about the interval's being closed, open, or half-open). In the limiting case of a = b, the interval is called "degenerate" with length zero (in this case we have a set containing a single point) (cf. Grünbaum, 1952:296).

Furthermore, division as an operation is only defined on sets and not on their elements, implying that the *divisibility* of finite sets consists in the formation of proper non-empty *subsets* of these (surely, the degenerate interval is indivisible by virtue of its lack of a subset of the required kind) (1952:301). Finally, the following two propositions are asserted and are considered to the perfectly consistent:

"1. The line and intervals in it are infinitely divisible" and
"2. The line and intervals in it are each a union of indivisible degenerate intervals" (1952:301).

If we confront this analysis of Grünbaum with our characterization of the nature of the actual infinite (the at once infinite), we soon realize that his whole approach is *circular*. We have seen that, on the basis of the regulative hypothesis of the at once infinite, not only the set of real numbers but also the number of line segments having a common end point could be considered as *non-denumerable infinite totalities*. In the latter case (i.e., in the case of a group of line segments), we may identify, within the modal structure of space, a retrocipation to an anticipation (a mirror-image of the structure of the system of rational numbers). This retrocipation to an anticipation ultimately underlies Grünbaum's statement: "the Cantorean line can be said to be already actually infinitely divided" (1952:300).

Seemingly, the objection that any denumerable sum of degenerate intervals (with zero-length) must have a length of zero, does not invalidate Grünbaum's claim that a positive interval is the union of a continuum of degenerate intervals, because in the latter case we are confronted with a non-denumerable number of degenerate intervals – obviously, if we cannot enumerate them, we cannot *add* them to form their sum (for this reason, measure-theory also side-steps the mentioned objection, valid for the denumerable case). (Any attempted "addition" would leave out at least one of them.)

11 Aristotle and modern mathematics – infinity and continuity

In order to overcome the problems posed by Zeno's paradoxes, Aristotle introduced his notion of the potential infinite (cf. also *De gen. et corr*. 316 a 14 ff.). The *infinite divisibility* of a line is granted by him, with the qualification that this divisibility is only potential and can therefore never be carried through *actually*. Furthermore, Aristotle upholds that no moving object is "counting" while it moves, since then the objection of Zeno would have been valid: in order to traverse a finite spatial distance it is required to actually count an *infinite* sequence of numbers.

Aristotle's notion of division specified an important trait of continuity: when we divide a continuous distance into two halves, we have to take the same point *twice* – in the first instance as a starting-point and in the second instance as an end-point. To this he adds that the infinite number of halves thus obtained is never *actual* but only *potential halves* (*Phys*. 263 a 23 ff.).

Aristotle firmly opposes the acceptance of the *actual infinite*, mainly in terms of the following objections (cf. *Phys*. 204 a 20 ff., Metaf. 1066 b 11 ff., and *Metaf*. 1084 a 1 ff.):

(i) if the actual infinite is composed out of parts, everyone of these parts should taken by itself would be actual infinite, implying the absurdity that the *whole* is no longer prior to (cf. Pol. 1253 a 19-20) (or greater as) a *part*;

(ii) if the actual infinite is composed out of finite parts we would be able to perform the impossible by counting the infinite, or else there must exist transfinite (cardinal) numbers being neither even or odd (it turned out that both objections actually saw something essential of the actual infinite!). [Since Bolzano an infnite set is defined as one in which it is possible to establish a one-to-one correspondence between the set and a proper subset of it. Owing to the fact that communtativity does not hold for Cantor's transfinite ordinal numbers, one can characterize *omega* (ω) as being both even and odd, and as neither being even nor odd (cf. Cantor, 1962:178-179).]

Aristotle considers it impossible to explain the continuity of a straight line in terms of the (infinite) number of its points. If “that which is infinite is constituted by points, these points must be *either* continuous *or* continuously in contact with one another” (*Physica*, 231a29-31). Points, however, are indivisible (a point has no parts), while “that which exists between two points is always a line” (*Physica*, 231b8). According to Aristotle it is clear that “everything which is continuous is divisible into divisible parts which can be divided infinitely: since if it was divisible into indivisible parts, we would have the divisible and indivisible in contact since the limits of continuous things are one (i.e. the same – DFMS)” (*Physica*, 231b15ff.).[1]

In the 19th century the new arithmeticistic tendency aimed at an arithmetical definition of spatial continuity. Bernard Bolzano already illuminates this tendency in par.38 of his work on the paradoxes of the infinite (cf. Bolzano, 1921). He mentions the objection that the attempt to construct extension out of

1 Von Weizsäcker says that the domain in which figures are defined displays, when compared with natural numbers, the property of *continuity*. In this context he calls upon Aristotle's view of continuity: “Continuity is defined by Aristotle as that which could be divided endlessly in similar parts. The parts of a continuum cannot be counted; but one can measure *continua*” (1993:115).

parts which *themselves* are *not* extended will contain a hidden circle, but he believes that the problem disappears when it is realized that "each whole" exactly "has numerous properties absent in the parts" (1920:72). This is the first instance of claiming an "emergent" property flowing from the whole. M.J. White explicitly employs the term "emegrent" when he explains that when topology considers large classes of points *collectively* "continuity becomes an 'emergent' property" (1988:8, cf. pp.9, 12).

The criterion which Bolzano sets for a continuum is that "a continuum is present there, but also only there where a set (Inbegriff) of simple objects (of points ...) finds itself, which is situated in such a way that every single object has at least an environment (Nachbar) (of points – DFMS) in this set for every distance (Entfernung) however small" (1920:73).

Cantor criticizes the criterion used by Bolzano as *insufficient* since a set constituted e.g. by distinct *continua* (and therefore being as a whole *discontinuous*) would still be continuous in terms of Bolzano's definition (the end points of each distinct continuum would after all still contain further points of the particular set in an arbitrarily small environment – cf. Cantor, 1962:194).

Cantor declares that he has no other choice but to posit a "general purely arithmetical concept of a point-continuum" supported by the way in which he defines real numbers (1962:192). He defines a point-continuum as a *perfectly coherent* set. A set is *perfect* when every point of the set is a *limit-point* and when *all* limit-points of the set belong to the set. He calls "T a coherent point set, when for every two points t and t' of this set, at a given arbitrarily small number ε there are always a finite number of points $t_1, t_2, t_3, ..., t_n$ of T' present in multiple ways, so that the distances tt_1, t_1t_2, $...t_nt'$ are all together smaller than ε" (1962:194).

Comment: Cantor's definition of coherence concerns a *metrical characteristic* of the continuum. In modern topology, however, continuity is described in terms of *open* sets (abstracted from the characteristics of a metrical space – cf. Willard, 1970:16-19). P.S. Alexandroff defines the continuum as a *non-empty compactly coherent set* (1956:163ff., 201ff.). A set is compact if every infinite subset has at least *one* limit-point. (A point *x* is a limit-point of a set A when every environment of *x* contains at least one point of A different from *x*). This implies that a set in an Euclidean space is only compact if it is *delimited*. In terms of Alexandroff's definition an infinite straight line is therefore *not* continuous, while for Cantor it *is* (cf. also Meschkowski, 1967:55).

Dedekind follows the continuity of a straight line arithmetically by successively introducing new numbers:

> If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes necessary that the instrument R constructed by the creation of rational numbers be essentially improved by the creation of new numbers (namely irrational numbers – DFMS) such that the domain of numbers shall gain the same

> completeness, or as we may say at once, the same continuity, as the straight line (Dedekind, 1901:9).

On this foundation Dedekind describes his well-known concept of a *cut* which characterizes continuity: when “all real numbers break up into two classes U_1, U_2, such that every number a_1 of the class U_1 is less than every number a_2 of the class U_2 then there exists one and only one number by which this separation is produced” (1901:20). Dedekind's notion of a *cut* is dealt with in analysis textbooks in such a manner that the real number which brings about the division is greater than or equal to every element in the one set and smaller than or equal to all the elements of the other set (cf. e.g. Bartle, 1964:51).

Cantor himself refers to the relation which exists between his view of a perfect set and Dedekind's cut theorem (1962:194). G. Böhme strikingly shows how Cantor's definition of the continuum contains two stipulations which both meet the twofold Aristotelian definition of a continuum, namely *coherence* and a characteristic which ensures the existence of dividing points for *infinite division* (1966:309). Only allowing a Dedekind-cut at divisions, Böhme justifies his statement as follows:

> when a Cantorian continuum as such is divided into two by means of the indication of a point so that the one set contains those points which are in numerical value greater than or equal to the indicated point, while the other set contains those points of which the numerical value are smaller than or equal to the numerical value of the indicated point, both parts are again continuous. Such divisions are possible into infinity (due to the perfection of the continuum), and the parts are still coherent in the Aristotelian sense (i.e. their limit-points are the same) (1966:309).

This is a remarkable situation: the Cantor-Dedekind description of the continuum presupposes the use of the actual infinite (in particular by using the actually infinite set of real numbers), but nonetheless meets Aristotle's two requirements for a continuum – and this while Aristotle *explicitly* rejects the actual infinite and only recognizes potential infinity! Did Aristotle actually use the actual infinite *implicitly*, or is the Cantor-Dedekind definition in the last instance not purely *arithmetically* founded?

12 Aristotle and Cantor: an ‘intermodal’ solution

We have concluded the previous paragraph by raising the question: how is it possible that both employ the same criteria (infinite divisibility and taking any point of division twice [as a starting-point and as an end-point]), but fundamentally oppose each other in connection with the infinite?

This almost perplexing situation is immediately clarified in terms of our analysis of the intermodal coherence between number and space. We saw that the original spatial whole-parts relation is indeed founded on the numerical time-order of succession (displaying the primitive meaning of infinity as *endlessness*), explaining the infinite (endless) divisibility of spatial continuity (this feature manifests the retrocipation at the factual side of the spatial aspect to the law-side of the numerical aspect). If one restricts oneself to this original meaning of the spatial aspect (without, in addition, considering the opened up meaning of the foundational aspect of number), as Aristotle implicitly did (understandable in the light of the *geometrization* of Greek mathematics), then no meaning could be given to the notion of actual infinity – explaining the relative correctness in the claim (put forward by Aristotle, Kant and many others in the history of philosophy and mathematics) that a line is not divided in the sense of the actual infinite (at once infinite), but only divisible in the sense of the potential infinite (successive infinite). It is *relatively* correct, because the foundational meaning of number (analogically reflected in the multiplicity of successive spatial divisions) is not itself as yet opened up by the anticipatory hypothesis of the at once infinite (referring the meaning of number, under the guidance of our theoretical thinking, to the modal meaning of space).

But this is exactly what Cantor did – he effectively used this anticipatory meaning of number in his description of a perfectly coherent set of real numbers (taken in their linear order), without realizing, however, that implicitly his use of the at once infinite still presupposes the primitive and irreducible spatial order of simultaneity. But given this disclosed approach, it is no longer possible to exclude the actually infinite dividedness of a line in terms of its endless divisibility. The notion of *endless divisibility*, employing the number-concept of the successive infinite, may be deepened with the foundational aid of the opened up numerical use of the number-idea of actual infinity.[1] Consequently, the diverging approaches to continuity present in the thought of Aristotle and Cantor (in spite of their formal agreement concerning the two mentioned criteria for continuity) may be summarized as follows:

> Both characterize continuity, but each one chooses his own view-point – Aristotle's angle of approach is the spatial aspect (which does not necessarily need the at once infinite), whereas Cantor's perspective uses the numerical anticipation to space (an approach which does need the at once infinite)!

The irreducibility of the spatial time-order of simultaneity to the numerical time-order of succession is ultimately dependent on the irreducibility of the modal meaning of space to that of number. From this it directly follows that the spatial whole-parts relation, determined by the spatial order of simultaneity, is also irreducible – explaining why the typical *totality* character of the continuum reveals an unavoidable circularity in the attempted purely arithmetical 'definition' of continuity. In other words, the modal meaning of space, qualified by the primitive meaning-nucleus of continuous extension (expressing itself at the law-side as a simultaneous order for extension and at the factual side as dimensionally determined extension – with or without a de-

1 Perhaps Hegel could have abused this perspective in the following way: a continuous whole is the "absolute" *an sich*; in its infinite divisibility, it reaches out to its opposite, the discrete nature of number; and finally, in the at once infinite as an anticipation to space the absolute idea returned to itself – the infinite whole, in itself divided in the sense of actual infinity.

fined metric), not only implies that this meaning-nucleus of space is irreducible to number, but also that the spatial order of simultaneity at the law-side and the whole-parts relation at the factual side of the spatial aspect are ultimately irreducible.

Although he did not pay attention to the law-side of the spatial aspect (obviously because he did not dispose of an articulated meaning-analysis of the structure of number and space), we have seen that Paul Bernays does appreciate the irreducibility of the spatial whole-parts relation (the totality feature of spatial continuity) (Bernays, 1976:74).

> The property of being a totality "undeniably belongs to the geometric idea of the continuum. And it is this characteristic which resists a complete arithmetization of the continuum" ("Und es ist auch dieser Charakter, der einer vollkommenen Arithmetisierung des Kontinuums entgegensteht – 1976:74).

Laugwitz realized that the real numbers, in terms of Cantor's definition of a set, are still individually distinct and in this sense 'discrete'. According to him the set concept was designed in such a way that what is continuous withdraws itself from its grip, for according to Cantor a set concerns the uniting of well-distinguished entities, implying that the discrete still rules.[1] Although this objection actually shows that Laugwitz did not understand the difference between the successive and the at once infinite properly, in its own way it could be seen as an objection to the arithmeticistic claims of modern mathematics. In this regard Laugwitz implicitly supports Bernays's deeply felt reaction against the mistaken and one-sided nature of modern arithmeticism, expressed in his words:

> The arithmetizing monism in mathematics is an arbitrary thesis. The claim that the field of investigation of mathematics purely emerges from the representation of number is not at all shown. Much rather, it is presumably the case that concepts such as a continuous curve and an area, and in particular the concepts used in topology, are not reducible to notions of number (Zahlvorstellungen) (1976:188).

13 Concluding remark

Although continuity belongs to the core meaning of the spatial aspect, and in this role qualifies the retrocipatory analogies to number within the spatial aspect, the meaning of spatial continuity expresses itself in its coherence with the (foundational) meaning of number. The apparently contradicting positions of Aristotle and Cantor-Dedekind underscores this perspective, because

1 "Der Mengenbegriff ist von vornherein so angelegt worden, daß sich das Kontinuierliche seinem Zugriff entzieht, denn es soll sich nach Cantor bei einer Menge ja handeln um eine "Zusammenfassung wohlunterschiedener Dinge ... – das Diskrete herrscht" (Laugwitz, 1986:10). And on the next page we read: "So kommt man dazu, die Frage nach der Mächtigkeit der Menge der reellen Zahlen als 'Kontinuumproblem' zu bezeichnen. In dieser Auffassung wird der Unterschied zwischen Diskretem und Kontinuierlichem verwischt: Je zwei Teilpunkte sind wohl voneinander unterschieden, aber ihre Gesamtheit soll das Kontinuum repräsentieren; dieses würde also durch Diskretes dargestellt."

it highlights that the former explores a spatial point of view (merely requiring the successive infinite) whereas the latter approaches continuity from the perspective of the spatially disclosed and deepened meaning of number (which has to employ the at once infinite). At the same time this analysis enabled us to argue that modern mathematical set theory actually is a *spatially deepened* theory of number. Bernays correctly states (as quoted on page 21 above):

> The idea of the continuum is a geometrical idea which analysis expresses in terms of arithmetic (1976:74).[1]

Not realizing this spatially deepened character of set theory causes the vicious circle entailed in all attempts to argue for a complete reduction of space to number on the basis of a theory employing the numerical anticipation to space (by accepting the at once infinite). In opposition to the attempt of Greek mathematics to reduce number to space and the modern pursuit to arithmatize space, the third alternative presented here aims at acknowledging both the *uniqueness* and *mutual coherence* of number and space.

14 Literature

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1 We have mentioned that a topological description of continuity may abstract from any given metric.

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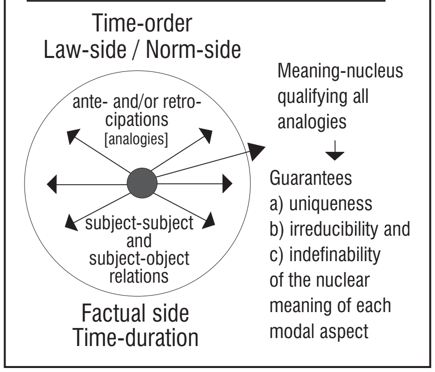
Sketches

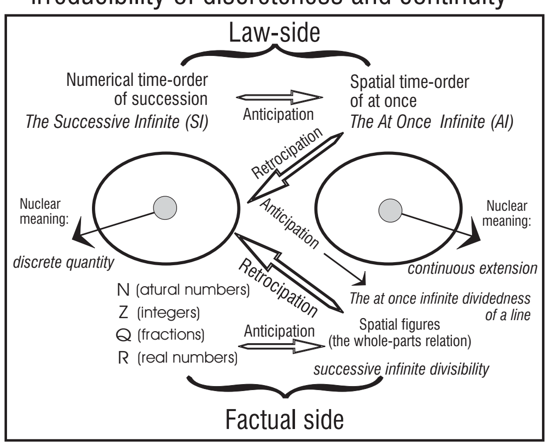
Sketch 1

The structure of a modal aspect



Sketch 2

The mutual coherence and irreducibility of discreteness and continuity



Index of Subjects

Index of Persons