The Concept of Number: Multiplicity and Succession between Cardinality and Ordinality[1]

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Abstract

This article sets out to analyse some of the most basic elements of our number concept – of our awareness of the *one* and the *many* in their coherence with multiplicity, succession and equinumerosity. On the basis of the definition given by Cantor and the set theoretical definition of cardinal numbers and ordinal numbers provided by Ebbinghaus, a critical appraisal is given of Frege's objection that abstraction and noticing (or disregarding) differences between entities do not produce the concept of number. By introducing the notion of subject functions, an account is advanced of the (nominalistic) reason why Frege accepted physical, kinematic and spatial properties (subject functions) of entities, but denied the ontic status of their quantitative properties (their quantitative subject function). With reference to intuitionistic mathematics (Brouwer, Weyl, Troelstra, Kreisel, Van Dalen) the primitive meaning of succession is acknowledged and connected to an analysis of what is entailed in the term 'Gleichzahligkeit' ('equinumerosity'). This expression enables an analysis of the connections between ordinality and cardinality on the one hand and succession and wholeness (totality) on the other. The conceptions of mathematicians such as Frege, Cantor, Dedekind, Zermelo, Brouwer, Skolem, Fraenkel, Von Neumann, Hilbert, Bernays and Weyl, as well as the views of the philosopher Cassirer, are discussed in order to arrive at an assessment of the relation between ordinality and cardinality (also taking into account the relation between logic and arithmetic) – and on the basis of this evaluation, attention is briefly given to the *definition* of an *ordered pair* in axiomatic set theory (with reference to the set theory of Zermelo-Fraenkel) and to the definition of an ordered pair advanced by Wiener and Kuratowski.

Introduction

We start our discussion with reference to the problem of the *one and the many* (multiplicity, plurality) and then proceed to incorporate the nature of succession in our reflections. Considering numerical relations within non-arithmetical contexts makes it

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possible, depending on the *unit of counting*, to come up with different 'countings'. For example, depending upon the *unit*, 'five chairs' may turn into 'twenty chair legs'.

Twentieth-century mathematics, with the aid of modern set theory, apparently succeeded in carrying the concept of number to a high level of abstraction by employing the idea of a one-to-one mapping.[2] I can remember that, as a first-year mathematics student, I read the work of Northrop on "Riddles in Mathematics," and that he explained this understanding of number by using the example of the leader of an expedition of forty-three people travelling in a country where the vocabulary of number words is limited to 'one', 'two', 'three', 'four', and 'many'. Northrop explained:

> Suppose further that you have gone on ahead to a village where you expect to spend the night, and that you are trying to make the village chief understand that you want food prepared for forty-three people. Assuming that he understands you want food, how will you put over the idea of 'forty-three'? Very likely you will do it by making marks on a piece of paper, or on the ground – one mark corresponding to each member of the party. If a plate of food corresponding to each mark is then prepared, you can be sure that each member of the party will be fed. Thus the chief, with no word whatever for the number 'forty-three', is able to *count* the number of people in the party, and the number of plates of food. To describe the process in somewhat more precise terms, you set up what is called a 'one-to-one correspondence' between the members of your party and the marks on the paper. The correspondence is 'one-to-one' because corresponding to each person there is one mark, and, conversely, corresponding to each mark there is one person. The chief then sets up a one-to-one correspondence between the marks and the plates of food (Northrop, 1964: 145).

This example illustrates the nature of what Cantor has called the *power* or *cardinality* of a set.[3] When no order relation between the elements of a set is taken into account, the *cardinality* of the set is intended – explaining the designation *cardinal numbers*. But when there is an order relation between the elements of a set, *ordinal numbers* are at stake.

Likewise Ebbinghaus explains a set theoretical definition of (the 'naive') natural numbers in terms of two different perspectives: (a) One can use them to *count*. The process of counting is thus constituted that one, starting with the number 0, repeatedly proceeds from a natural number *n* already reached to its *successor (n+1)*. It provides the basis for a *sequencing* or *ordering* of the natural numbers (their *ordinal aspect*). (b) One can use the natural numbers in order to specify the number of elements of finite sets (or, as it is said, their *power*) (*cardinal aspect*).[4] Maddy captures this difference succinctly: "Cardinal numbers tell 'how many' – one, two, three ... – as opposed

2 Cantor used the German phrases "gegenseitig eindeutig" and "eineindeutig" (Cantor, 1962:441, 444).

3 Cantor holds that the concept of a cardinal number emerges when, with the aid of our active thought-capacity, we abstract from the character of the different elements of a given set M and also disregard the order in which they are given (Cantor, 1962:282).

4 "Die naiven natürlichen Zahlen kann man unter zwei Aspekten betrachten: (a) Man kann mit ihnen *zählen.* Der *Zählprozeß* besteht darin, daß man, beginnend mit der Zahl 0, wiederholt von einer bereits erreichten natürlichen Zahl *n* zu deren *Nachfolger (n + 1)* übergeht. Er begründet eine *Reihenfolge* oder *Ordnung* der natürlichen Zahlen *(ordinaler Aspekt).* (b) Man kann die natürlichen Zahlen verwenden, um die *Anzahl der Elemente* (oder, wie man sagt, die *Mächtigkeit)* endlicher Mengen anzugeben (*kardinaler Aspekt*)" (Ebbinghaus, 1977:55).

to ordinal numbers, which tell 'how many-ith' – first, second, third, ..." (Maddy, 1997:17).

All seems fair and clear in this distinction between cardinality and ordinality – in the differentiation between *cardinal numbers* and *ordinal numbers*[5] – at least as long as one does not ask the question which of the two notions is more fundamental than the other.

According to Hallett (1984:50-51), Cantor regarded his transfinite numbers basically as ordinal numbers, a view shared by Dummett where he claims that, if Frege did pay enough attention to Cantor's work, he "would have understood what it revealed," namely "that the notion of an ordinal number is more fundamental than that of a cardinal number" (Dummett, 1995:293).[6] It does therefore appear as if the difference between Frege and Cantor is that the former proceeds from cardinal numbers and the latter from ordinal numbers.

But Tait convincingly argues that in the *Grundlagen* (of 1883), Cantor does not identify numbers with ordinals, because "Cantor's definition of the numbers stands on its own feet and is entirely independent of their application e.g. as measures of well-ordered sets" (Tait, 2005:257).

The one and the many

Frege actually struggled with a distinction he has not acknowledged, namely that between the acquisition of (general) concepts through abstraction (*entitary-directed abstraction*) and that of lifting out universal aspectual frameworks within which concrete entities function (to be designated as *modal* or *aspectual abstraction* – see Strauss, 2003:72-74). The lack of this distinction is connected with his misunderstanding of the ambiguity entailed in the German word "gleich" – meaning either *equality* or *identity* (see Tait, 2005:239, 242). This led Frege to argue against the idea that the concept of number could be attained through any process of abstraction.

In the absence of a theory of aspects of reality, Frege did not contemplate the nature of *modal abstraction* in connection with the number concept. When a multiplicity of concretely existing things (or entities) also exhibit a function within the (universal) structure of the quantitative mode of reality, the decisive issue is not whether, through (entitary directed) abstraction, higher level (type) concepts are formed, but rather whether the core meaning of number (discrete quantity) applies to (any) multiplicity of entities functioning within this aspect. Number in the first place concerns the *multiplicity* of entities, and not the entities *as* entities. What is shared by a multiplicity of things, is the quantitative property of *being distinct*. Even if Frege may have it his way in arguing about abstracting from the typical differences between entities, this line of argumentation does not account for the way in which entities function within the framework of the quantitative aspect. And he seems to have sound arguments against this kind of approach. If the term 'unity' is meant to designate "objects to be counted" then, so Frege argues, one cannot define number as (multiple) 'unit(ie)s' (*Einheiten*). In general he claims that "a plural is only possible of concept-words."

A crucial part of Frege's argument is that, as long as different entities are brought to-

5 The standard German practice is to refer to "Zahl" and "Anzahl" – although we shall see that in this regard a serious misunderstanding prevailed between Frege and Cantor.

6 Dummett continues by pointing out that this is also true of the finite case: "after all, when we count the strokes of a clock, we are assigning an ordinal number rather than a cardinal. If Frege had understood this, he would therefore have characterised the natural numbers as finite ordinals rather than as finite cardinals" (Dummett, 1995:293).

gether, the differences between them remain intact – and number does not constitute these differences. But note that these differences are typical differences, i.e. differences reflecting the fact that we are dealing with entities. He argues that, when one abstracts from the peculiarities of individuals in a "collection of objects," or when one disregards the properties that distinguish separate things, then it is not the case that what is left is the concept of their number (*Anzahl*), for instead one arrives at a general concept embracing every one of these things. In other words, the outcome is, according to Frege, one general type concept.

Against this background, Frege considers two equally unacceptable options. When we want to generate number through the collection of different entities, we obtain an accumulation in which the objects with precisely those properties through which they are distinct are found – and that is not number. When, on the other hand, we want to construct number through the union of what is the same (*gleich*), everything always collapses into one – in which case we are unable to arrive at the many (*Mehrheit*). Whenever no differences are left, multiplicity shrinks into oneness.

Frege's argument is burdened by two mistakes. (i) The first is the focus on *typical differences* instead of *quantitative distinctness,* and (ii) the second is found in his (above-mentioned) inability to distinguish properly between modal abstraction and entitary directed abstraction – to be seen in his lack of distinguishing between modal equality and entitary identity ("gleich" in the sense of "similar" and "gleich" in the sense of "identical").

Re (i): It is certainly correct to claim that number (a modal, functional concept) does not constitute the (typical) differences between entities, but this insight does not warrant the conclusion that the multiplicity (plurality) of entities involved in such a process of (entitary-directed) abstraction is cancelled by this very process. This consideration naturally leads to our second point.

Re (ii): Frege first of all summarises the outcome of his argumentation by pointing out that number is not a property that can be abstracted from things in the way in which colour, weight and density could be abstracted from such things – which leaves us with the question "about what is something said in the specification of a number"?

Frege holds that number is not something physical, nor is it something subjective, like a representation – just as little as it is the union of one thing and another thing. Expressions like multiplicity, set, and plurality are indeterminate and therefore not appropriate to serve as an explanation of number. Features like being delimited, being undivided, and being unanalysed are not useful for what we express with the word 'one' either. If the things to be counted are called 'Einheiten' (unities), then the unqualified assertion that these 'Einheiten' are the same ('gleich'), is false. That they are similar in a specific respect is indeed true, but useless. It seems as if we have to assign to the 'Einheiten' two contradictory properties: identity and difference.

The two words "identity" and "difference" implicitly appeal to the dimensions of entities and aspects: different instances of one or other kind of entity are subsumed under the general (universal) entity-concept involved, and Frege correctly argues that this concept is *one*. Yet in our everyday experience, counting always concerns both dimensions, for the first (entitary directed) question is: what is counted? whereas the second pertains to the (modal) quantitative question: how many are there? Every specimen counted (or: countable) is similar ("gleich") to every other one in the numerical sense of just being another 'one' to be counted. An entitary perspective on the previous sentence yields the unified "what-concept," while a modal functional arithmetical perspective generates the “number concept” of how many of these entities there are. In other words, the conceptual identity of a multiplicity of entities cannot eliminate this (modal, functional) numerical multiplicity![7]

Although he is not acquainted with the distinction between the entitary dimension and the dimension of modal aspects, Tait has a very clear understanding of the above-mentioned states of affairs. He claims that Frege tends to confuse the two questions: “What are the things to which number applies? And, what are numbers?” (Tait, 2005:241). In order to highlight the difference between the what and the how (i.e. the difference between entities and aspects), we preferred to phrase the second question in “how” terms by asking “how many are there,” instead of “what are numbers.”

Tait also points out the confusion in Frege's thought regarding the meaning of the word “gleich”: “His reading seems to me to have been misdirected by two related things: his interpretation of 'gleich' to mean 'identical' and his failure to understand the historical use of the term 'number' to mean what is numbered” (Tait, 2005:242).

A theoretically articulated understanding of the meaning of number is therefore only possible on the basis of modal abstraction (denied by Frege) – the theoretical activity through which a specific aspect is lifted out while disregarding others.

This account of number as something related to *concepts* actually uses (logical) concepts as a substitute for the quantitative aspect of reality. Worded differently, one can say that Frege denies the *ontic status* of the quantitative domain of reality by replacing it with the domain of *logical* (conceptual) *objectivity*.

In order to explain what has just been said, we may take an *abacus* with plastic pieces as an example. Although the purpose of an abacus is to aid children in their mastering of *arithmetical* relations (and operations), it functions with a striking *many-sidedness*. Children immediately perceive the different colours, the spherical shapes and the possibility to move the distinct pieces left and right. This is made possible by the fact that the abacus *actively functions* in the physical aspect (plastic pieces), in the kinematic aspect (moving the pieces back and forth), as well as in the spatial aspect (the shape of the pieces).

It would be more correct to say that the abacus has *subject functions* within these three aspects of reality. This mode of speech entails a critique of the subject-centered manner in which philosophers normally refer to things like abacuses by designating them as *objects*. As a physical (material) entity, an abacus is subject to physical laws, implying that it is a *physical subject* and not a *physical object*. Likewise, as a spatial and kinematic subject, the abacus actively functions within the spatial and kinematic aspects. But within aspects such as the latent sensitive mode (as something perceived), the logical-analytical (as something conceived), the sign-mode (as something named or designated), the same physical subject (such as an abacus) functions as an *object* –

7 After the introduction of set theory, it was realised that it cannot be defined in a non-circular way. Wang remarks that the “theories of real numbers by Dedekind, Cantor, and others are insufficiently explicit because they presuppose but do not include a theory as to how sets are to be introduced” (Wang, 1974:76). This explains why the axiomatic foundation of Zermelo Fraenkel's set theory introduced the membership relation as a primitive (undefined) term (see Fraenkel *et.al*., 1973:22 ff.). Gödel explicitly argues: “The operation 'set of x's' (where the variable 'x' ranges over some given kind of objects) cannot be defined satisfactorily (at least not in the present state of knowledge), but can only be paraphrased by other expressions involving again the concept of set, such as: 'multitude of x's', 'combination of any number of x's', 'part of the totality of x's', where a 'multitude' ('combination', 'part') is conceived of as something which exists in itself no matter whether we can define it in a finite number of words (so that random sets are not excluded)” (Gödel, 1964:262).

dependent upon and correlated with a *subject function*. Animals and human beings can *open up* or *make patent* the sensitive object function of an abacus by perceiving (and thus objectifying) it. Human beings can acquire the concept of an abacus by distinguishing it from whatever is *not* an abacus. By doing this they disclose the *analytical object function* of the abacus. Opening up, disclosing or deepening the object functions of physical entities is always the result of the activity of a *subject*. In other words, insofar as *material things* are *physical*, they are not *objects* but *subjects* – and insofar as they are *objects*, they are no longer understood according to their physical subject function, for now they are viewed according to their non-physical properties (i.e. their post-physical *object-functions*).

Yet our account of the subject functions of the abacus still lacks an important aspect, namely the *quantitative facet*. Was it not 'natural' to expect that Frege would accept, alongside the acknowledgement of *physical* properties, *kinematical*, *spatial*, and also *quantitative properties* (of entities like an abacus)? Since the early modern era, it was not strange at all to acknowledge at least the first three subject functions of material things, namely the arithmetical, spatial and kinematic properties of matter. As primary qualities of matter, Galileo considers arithmetical (countability), geometrical (form, size, position, contact) and kinematical properties (movement).[8]

A brief historical remark may help us understand what happened here. Since the Renaissance modern *nominalism* denied any *universality* outside the human 'mind' – as Descartes emphatically stated "number and all universals are modes of thought" (*Principles of Philosophy*, Part I, LVII). Also Gauss believed that number is merely a product of the human mind.[9] In this way reality is stripped from its quantitative function. But then something else has to take its place – and in the case of Frege it was taken by (objective) *concepts*, which became the 'bearers' of the former ontic property of *numerosity*. Concretely existing entities no longer have a numerical subject function; they do not display a quantitative aspect alongside their spatial, kinematical and physical functions – for Frege this numerical subject function is replaced by a "conceptual subject function"![10]

Of course an account of the concept of number *is* complicated. Hao Wang remarks that the famous mathematician, Kurt Gödel, is very "fond of an observation that he attributes to Bernays": "That the flower has five petals is as much part of objective reality as that its color is red" (quoted by Wang, 1982:202). Bernays does not want to accept the "concrete" as the only reality: "It appears that it is only a preconceived philosophical view determining this requirement, the view namely that only one kind of factuality could exist, that of concrete reality" (Bernays, 1976:122). Phrased in terms of our suggested distinctions, Bernays here refers to the *ontic status* of quantitative properties, i.e. to an acknowledgement of the *arithmetical aspect of reality*. Gödel struggled in his own way with this ontic dimension of functions or aspects, for he wanted to ascribe an 'objective' status to the field of investigation of mathematics, ac-

8 See Galileo (1957:274) and Hucklenbroich, who writes: "G. Galilei zählt als primäre Qualitäten der Materie arithmetische (Zählbarkeit), geometrische (Gestalt, Größe, Lage, Berührung) und kinematische Eigenschaften (Beweglichkeit) auf" (Hucklenbroich, 1980:921).

9 By contrast, Gauss thought that space and time had a reality outside the human mind (see Kronecker, 1887:265).

10 Since the expression "subject function" here has a structural meaning – and therefore not the sense of "merely produced by a subject" – it is consistent with the (above-mentioned) "objective" status given to a concept by Frege. In terms of Frege's position one should here rather speak of an "objective conceptual function."

cessible by means of what he labeled *semiperceptions* and representing "an aspect of objective reality" (quoted by Wang, 1988:304).[11]

Having restored an acknowledgement of the aspectual nature of number (in its *ontic* sense) we may now focus on multiplicity in the sense of *being distinct*, and its connection with the primitive meaning of succession.

Falling back onto 'being distinct' and 'succession'

As soon as Frege embarks upon the challenge to account for "der Begriff der Anzahl" (the concept of *number*), his entire argument demonstrates the circularity involved in any attempt to explain *succession*[12] – every explanation begs the question. He circumscribes the number 0 as follows: "the number 0 is ascribed to a concept F when in general, whatever a may be, the proposition is valid that a does not fall under this concept".[13] In a similar way the number 1 is explained: "The number 1 is ascribed to a concept F when it is not generally valid that whatever a may be the proposition holds that a does not fall under F, and when it generally follows from the propositions 'a falls under F' and 'b falls under F' that a and b are the same" (Frege, 1884:67). This explanation simply proceeds on the implicit awareness of the *distinctness* between 'nothing' and 'something' in their coherence with the logical act of *negation* and *affirmation*. But such a logical act presupposes the primitive meaning of a *discrete multiplicity* (the one and the many) – as an *ontic* function.

Frege does concede that in the language of ordinary life, number appears *attributively* – but given his denial of the *ontic nature* of the *quantitative mode*, he has to reformulate such statements. Instead of saying "Jupiter has four moons" one should say: "The number of Jupiter's moons is four" (Frege, 1884:69). Frege employs the idea of the 'scope' of a concept ("*der Umfang des Begriffes*").[14]

According to Frege, one cannot represent number as an independent *Gegenstand* ('object'), since it is neither something sensory, nor a property of an external thing. This becomes most clear in the case of the number 0, for it is impossible to attempt to imagine 0 visible stars.[15] Frege is correct in claiming that *number* is not something *sensory*, that it cannot be perceived by any of the senses, but acknowledging the *modal universality* of the ontic modes of reality does not collapse the *quantitative aspect itself*, in its *modal universality*, into a mere *property* of something. The ontic modes belong to a unique dimension of the universe that lies at the basis of the concrete functions of entities (and events). Therefore the modal properties displayed by the latter are

11 Although Wang feels "that 'an aspect of objective reality' can exist (and be 'perceived by semi-perceptions') without its occupying a location in spacetime in the way physical objects do," he does not dispose over a theory of ontic aspects or modal functions (see Wang, 1988:304). Dummett speaks of the "non-logicist platonism" of Gödel (cf. Dummett, 1995:301) – see also Strauss (2003:69-72).

12 In passing, we may note that the meaning of succession turned out to be crucial for an avoidance of antinomies in the development of set theory. For this reason the "central motivating concept is the *iterative concept*" in the axiomatisation of Zermelo (see Maddy, 1997:48).

13 "... einem Begriffe kommt die Zahl O zu, wenn allgemein, was auch a sei, der Satz gilt, daß a nicht unter diesen Begriff falle" (Frege, 1884:67).

14 "The number ascribed to a concept F is the scope of the concept 'equinumerous with the concept F' " ("Die Anzahl, welche dem Begriffe F zukommt, ist der Umfang des Begriffes »gleichzahlig dem Begriffe F«" (Frege, 1984:80).

15 "Die Zahl kann weder als selbständiger Gegenstand noch als Eigenschaft an einem äußern Dinge vorgestellt werden, weil sie weder etwas Sinnliches noch Eigenschaft eines äußern Dinges ist. Am deutlichsten ist die Sache wohl bei der Zahl 0. Man wird vergebens versuchen, sich 0 sichtbare Sterne vorzustellen" (Frege, 1884:70).

made possible by the universal structure of the different modal aspects – and the latter *transcends* any particular entity with its (typically specified) modal properties. Since the scope of a modal aspect encompasses every possible concrete entity, modal laws hold *universally* (such as functional quantitative laws, modal spatial laws, modal kinematic laws, modal physical laws, and so on). By contrast, the laws for particular *types of entities* are *specified* and consequently hold for *limited classes of entities* only.[16]

Both the arithmetical properties of entities and the concept of number therefore finds their ontic foundation in the co-conditioning role of the modal universality of the quantitative mode of reality.

Of course neither Frege nor any other mathematician can side-step the basic nature of *multiplicity* and *succession*. This becomes all the more clear when Frege sets out to provide a general explanation of the "transition from a number to the next one."[17] It is evident that without a *prior* insight into the *primitive meaning* of a *distinct* "nächstfolgenden Zahl" (a *successor*), one cannot even begin to explain the meaning of succession – thus making every such attempt circular.

The numerical time-order of succession

Kant already distinguished between *succession* and *causality* – for although *day* and *night* succeed each other, it is meaningless to say that the day is the *cause* of the night or vice versa. What is more remarkable is that the history of Western philosophy reveals an insight into different *modes of time* without having been able to relate it to a general theory of functional modes of time. In spite of his general (one-sided *psychologistic*) view that time is a(n internal) form of intuition, Kant distinguishes between *three* 'modes' of time: "The three modes of time are *endurance*, *succession* and *simultaneity*."[18] This remark matches our above-mentioned distinction between number (*succession*), space (*simultaneity*) and motion (*constancy*).[19] Within the quantitative mode of reality the basic order of succession is indeed a *time-order of succession determining* every possibility, to discern whatever is successive. Early in the 19th century, W. Hamilton even defined algebra as the "science of pure time or order in progression" (Hamilton, 1833 quoted by Cassirer, 1957:85). When this quantitative time-order of succession is highlighted by means of *modal abstraction*, i.e. by means of *identifying* and *distinguishing* within the framework of the universal modal structure of the various functional aspects of reality, its *reversibility* is immediately evident. By merely changing the sign, one can reverse the succession of integers in the plus and minus directions.[20] Yet we shall argue below that the time-order of succession alone does not provide a sufficient foundation for distinguishing between cardinal and ordinal numbers.

The time-order of succession belongs to the quantitative domain of the one and the

16 *Specified* does not mean *individualised*, because *universality* and *individuality* are irreducible and not found at the two ends of a continuum. The law for atoms has its own universality – it holds for *all* atoms – but this universality is *specified*, for not *everything* in the universe is an atom.

17 "Es bleibt noch übrig, den Übergang von einer Zahl zur nächstfolgenden allgemein zu erklären" (Frege, 1884:67).

18 "Die drei modi der Zeit sind Beharrlichkeit, Folge und Zugleichsein" (Kant, 1787:219).

19 Compare also the three modes distinguished by Galileo (note 15, p.6 above).

20 Since the operations of *addition*, *multiplication* and *subtraction* are *closed* over the set of (positive and negative) integers, the number zero does not create a problem – it has its place between the positive and negative integers (any integer subtracted from itself yields 0).

many, of distinctness and multiplicity, having as its core meaning the intuition of a *discrete quantity*.

The numerical order of succession in Brouwer's intuitionism

Brouwer believes that mathematics is ultimately founded on an unlimited succession of signs (or a finite sequence of signs), determined both by a first sign and by a law that derives the successor of every one of these sequences of signs (Brouwer, 1925:244). Similarly, a set (*Menge*) is a law (*Gesetz*), on the basis of which, amongst other possibilities, every arbitrary choice of a number generates an unfinished series of signs (Brouwer, 1925:244).

In modern mathematics, in general, we have two opposite claims. Brouwer, on the one hand, abandoning Kant's doctrine of the apriority of space, at the same time attempts, on the other hand, to strengthen Kant's doctrine of the apriority of time. Brouwer explains: "This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, in as much as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely" (Brouwer, 1964:69). In his Ph.D. dissertation, Brouwer introduced the "oerintuitie" (arch-intuition) of mathematics as the intuition of "multiple-unity" ("veel-eenigheid") (Brouwer, 1907:98).[21] Clearly, the core of this view is given in its dependence on what we have called the numerical time-order of succession. But the intuitionistic approach has an inherent ambiguity in this regard. The notion of infinity, on the one hand, is viewed as a law or principle (intimately related to induction – cf. Brouwer, 1964:71 and Weyl, 1926:23),[22] and on the other it is seen as a subjective and constructive process (then even constituting the "law side"). This ambiguity is caused by the lack of carefully distinguishing between the ontic nature of the arithmetical function of reality[23] and the subjective theoretical analysis and disclosure of its modal meaning. The latter activity presupposes the underlying structure of the numerical mode and can therefore never be constructed in a *purely subjective* way. Nevertheless, even the notion of natural numbers such as one and two, is the outcome of subjective human reflection (either non-theoretical or theoretical) on the primitive meaning of an arithmetical multiplicity (in succession). In this respect, intuitionism is justified in stressing the fundamental and *primitive meaning* of succession – Weyl correctly claimed that the essential character of the natural numbers, expressed in the feature of one, another one, "and so on," cannot logically be reduced to something more primitive (Weyl, 1926:11). This makes it clear why the hierarchy of types developed by Russell in his *Principia Mathematica* cannot "be described without taking resort to the intuitive concept of iteration" (Weyl, 1946:8). It furthermore elucidates the priority given to the primary notion of an ordinal

21 In 1919, this designated as the *time-intuition* or the intuition of two-oneness (twee-eenigheidsintuitie) (Brouwer, 1919:14).

22 Since intuitionistic mathematic restricts itself to the potential infinite, it rejects the application of the principle of the excluded middle in the case of the infinite – see Strauss (1991).

23 Just recall what has been said earlier about Gödel, Bernays and Wang regarding the ontic status of the numerical aspect of reality.

number in intuitionism (cf. Weyl, 1921:40, 43, 57, 58, 67; 1966:53), and the rejection of Cantor's notion of cardinal numbers (cf. Weyl, 1921:67).[24]

Interestingly, Brouwer eventually moved beyond his initial 1925 position, where a set is understood in term of a *law*. In 1952 he introduces the empty (languageless) form of 'two-ity', divested of all quality as the "basic intuition of mathematics" (Brouwer, 1952:141). The second act of intuitionism creates "the possibility of introducing the *intuitionist continuum* as the species of the *more or less freely proceeding* convergent infinite sequences of rational numbers."[25] The intuitionistic emphasis on the numerical order of succession underlies the development of the idea of "infinitely proceeding sequences" and inspired a wealth of intuitionistic investigations in what was soon called (the theory of) *lawless* sequences (see Kreisel, 1968; Troelstra, 1969 and Van Dalen & Troelstra, 1970).

Equinumerosiy and being distinct

Before we proceed, we once more briefly have to return to the (above-mentioned) conviction of Frege, namely that one either ends up with a single (general concept) or with the existing distinct entities in their fully nuanced nature prior to any 'abstraction'.

If the function of a collection of entities within the spatial aspect is given in their peculiar shapes and sizes, the question may be how should we conceive of the function of these entities within the quantitative mode of reality? That is to say, what happens if we look at them from the perspective of the quantitative mode? Clearly, from this angle of approach, we do not discern their strength, their relative movement or their form, but solely their *quantity*. How *many* are there? This question has disregarded all non-arithmetical properties by focusing only on the quantitative side of these things. Hume already accounted for this state of affairs by employing an idea of *equivalence*: "When two numbers are so combined, as that the one has always an unit answering to every unit of the other, we pronounce them equal" (Hume, 1739:117 – original spelling). Frege refers to *gleichzahlige Begriffen* (his notion of *Anzahl*), while Cantor introduces his idea of a cardinal number as the *invariant* shared by all equivalent sets (*Mengen*).

Weyl, also in this context, articulates the idea of *invariant* properties and relations ("invariante Eigenschaften und Beziehungen") in order to account for the notion of cardinal number (*Anzahl*): "Two sets *A* and *B* of objects ... are called equi-numerous (*gleichzahlig*), $A \sim B$, when it is possible to establish a one-to-one correspondence (*beiderseits eindeutig*) between the elements of *A* and the elements of *B*. ... This equinumerosity (*Gleichzahligkeit*) is obviously an equivalence [relation]. 'Every set determines a number; and two sets determine the same number if and only if they are equinumerous'."[26] And he does realise that this formulation is no longer subject to the criticism of Frege. He says that, in an inadequate way, one tends to express this by

24 We should note that Weyl here acknowledged the finite cardinals, referred to as the "number concept of everyday life" (Weyl, 1921:68), but rejected Cantor's transfinite cardinals as "mathematically useless" (Weyl, 1921:68). See also Van Stigt, 1990:147 ff.

25 See Brouwer, 1952:142. In a footnote on this page, Brouwer points out that the intuitionist concept of the continuum does not depend on the concept of a *measure*, for it can be "spread over an arbitrary fundamental sequence which has been completely ordered as an everywhere dense species with a first and a last element, and has been provided with a definition of convergence based exclusively on the relations constituting its everywhere dense order."

26 "Zwei Mengen *A* und *B* von Gegenständen (etwa die Personen und Stühle in einem Saal) nennen wir gleichzahlig, $A \sim B$. wenn es möglich ist, die Elemente von A mit den Elementen von *B* beiderseits eindeutig zu paaren (wenn es möglich ist, auf jeden Stuhl eine Person zu setzen, so daß kein Stuhl frei

saying that the number of a set is generated when one abstracts from the nature of its elements and solely hangs on to their distinctness (*Unterschiedenheit*). The objection sometimes fomulated, namely that when they are degraded into mere unities all elements will coincide in one, is avoided by the above-mentioned precise formulation.[27]

Gleichzahligkeit: the interplay of 'succession' and 'simultaneity'

The term *Anzahl* used by Frege actually intends to designate exactly what Cantor had in mind with his notion of a cardinal number (*Kardinalzahl*). Yet Cantor only partially succeeded in doing justice to the views of Frege in a review article, because he did not realise that underneath the terminological differences converging views were hidden. Zermelo finds it sad that the contemporaries, Frege and Cantor, the eminent mathematician and the meritorious logician, had such a bad mutual understanding of each other's work (remark in Cantor, 1962:442).

We have made the general observation that cardinal numbers relate to the *quantitative* question: *how many*? Ordinal numbers by contrast, account for the order relation between numbers. The *axiomatic-formalist* and *logicist* trends in modern mathematics (Hilbert, Gödel, Russell and others) apparently tend to affirm the primacy of cardinal numbers, while the intuitionist school (Brouwer, Weyl, Heyting and others) give preference to the foundational role of ordinal numbers.

Two issues ought to be discussed: (i) the nature of ordinality and cardinality in their relation to *succession* and *simultaneity*, and (ii) the relationship between *logical analysis* and *quantity*.

Ordinality and cardinality in their relation to succession and simultaneity

The fact that cardinal numbers disregard any order relation between the members of a set may suggest that, from the perspective of axiomatic set theory the concept of cardinal number appears to be more general and therefore more fundamental than the concept of ordinal number. Yet modern set theory indeed realised that ordinality is the primary concept. This present-day position finds support in the orientation taken by some earlier thinkers. For example, according to Smart, the main purpose of Cassirer's "critical study of the history of mathematics is to illustrate and confirm the special thesis that ordinal number is logically prior to cardinal number, and, more generally, that mathematics may be defined, in Leibnizian fashion, as the science of order" (Smart, 1958:245).

We have seen that the idea of *equinumerosity* (*Gleichzahligkeit*) is crucial for the concept of a cardinal number. Let us therefore commence by investigating what is entailed in this notion of *Gleichzahligkeit*.[28] First of all its articulation implicitly makes an appeal to *wholeness* or a *totality* (a *collection* or a *set*).

Cassirer relates the transition from ordinal to cardinal numbers explicitly to the difference between a *mere succession* of numbers and a specific *multiplicity*:

bleibt, aber auch jede Person einen Platz bekommt). Die Gleichzahligkeit ist offenbar eine Äquivalenz. 'Jede Menge bestinunt eine *Anzahl;* zwei Mengen dann und nur dann dieselbe Anzahl, wenn sie gleichzahlig sind' " Weyl (1966:24).

27 "In nachlässigerer Form pflegt man das so auszudrücken, daß die Anzahl aus der Menge entsteht, wenn man von der Natur ihrer Elemente abstrahiert und lediglich ihre Unterschiedenheit festhält. Der zuweilen gemachte Einwurf, daß alle Elemente, wenn man sie zu bloßen Einsen degradiert, in eines zusammenfielen, wird durch die obige präzise Formulierung abgeschnitten" (Weyl, 1966:24).

28 The term 'equinumerous' is still used by contemporary mathematicians in their discussion of the nature of cardinality – see for example Chihara, 2004:177.

> As soon as we proceed from the mere succession of numbers to a specific multiplicity (*Vielheit*), we encounter the transition from ordinal numbers to cardinal numbers as developed by Dedekind, Helmholtz and Kronecker (Cassirer, 1910:53).

A *multiplicity* in this sense is *given at once*, *simultaneously*. Without this moment of *simultaneity*, the key element of *comparing* sets – namely the idea of a *one-to-one-correspondence* – looses its meaning. Furthermore, also the general set theoretical thesis, that *when* two sets entail a one-to-one mapping of their elements ("gegenseitig eindeutig aufeinander abbildbar [sind]"), they are *equivalent* to each other, depends on this *order of simultaneity*. Cantor intuitively stated this theorem without providing proof.[29] Although the comparison of arbitrary cardinalities looks fairly simple, its proof in fact turned out to be much more complex, for it can only be proven with the aid of the idea of well-ordering.[30] The *comparison* of *cardinalities* therefore entails both *succession* and *at once* (*simultaneity*). Considering the cardinality of a set requires the presence *at once* of all its elements, although any attempt to *investigate* these elements themselves needs to recognise *some or other* order of succession. But if all the elements of a (finite or infinite) set are given at once, then the concept of a cardinal number unbreakably coheres with the idea of a *totality*. This insight immediately follows from the significant term used by Frege, Skolem, Weyl and Cassirer in this regard: *Gleichzahligkeit* (see Cassirer, 1910:58). Skolem does point out (calling upon Zermelo) that ordinal numbers and cardinal numbers concern the *comparison* of sets (see Skolem, 1929:80). But the crucial question is: what is required in order to compare sets? How is it possible to establish the *equivalence* of two cardinal numbers?

Cantor's set theory does this through the employment of his conception of a "one-by-one" *mapping* of the members of two sets. Evidently, such a mapping presupposes an order of *succession*, because otherwise no one-to-one correspondence could be acknowledged. In other words, *comparing cardinals* presupposes some or other *order of succession*! With reference to the concept of equivalence, Cassirer furthermore highlights the distinction between *how many* and *just as many*. A straightforward *multiplicity* (how many / *Wieviel*) differs from the concept "just as many" / "equalling" (*Gleich-viel*) (see Cassirer, 1910:62). He also emphasises that the "determination of number by the equivalence of classes presupposes that these classes themselves are given as a plurality" (Cassirer, 1923:52). This view is linked to his awareness of a "new logical function" in the "formation of the cardinal number":

> As in the theory of ordinal number the individual steps are established and developed in definite sequence, so here the necessity is felt of comprehending the series, not only in its successive elements, but as an ideal whole (Cassirer, 1953:42).

The crucial question at this point is whether or not the intuition of a *whole* (*totality*), which is given *at once*, stems from our quantitative intuition of *succession* or whether it has an *irreducible meaning* transcending the confines of quantitative relations? In

29 E. Schröder (in 1896) and F. Bernstein (in 1897) have first proven this *equivalence theorem* (see Zermelo in Cantor, 1962:209, note 5). See also Zermelo (1908:267 ff.)

30 Cantor defines a set (*Vielheit*) as being well-ordered when it meets the condition that every subset (*Teilvielheit*) has a *first* element – such a set is summarily designated as a succession (*Folge*): "Eine Vielheit heißt wohlgeordnet, wenn sie die Bedingung erfüllt, daß jede *Teilvielheit* ein *erstes* Element hat; eine solche Vielheit nenne ich kurz eine 'Folge' " (Cantor, 1962:444).

other words, is it possible to bring *succession* and *simultaneity* under the same *denominator* or are they *irreducibly different*?

That *simultaneity* and *wholeness* transcend the primitive meaning of number could be argued with an appeal to the views of Bernays and Skolem. Paul Bernays (the co-worker of David Hilbert) points out that mathematical analysis deals with *the conceptual clarification of geometrical representations* and that *the totality-character of space* obstructs a *complete arithmetisation of the continuum* (see Bernays, 1976:VIII; 74).[31] Add to these perspectives Gödel's remark regarding the nature of *sets*. Although modern (axiomatic) set theory (Zermelo, Fraenkel, Hilbert, Ackermann, Von Neumann) pretends to be a purely (atomistic) *arithmetical* theory, the structure of set theory actually implicitly (in the undefined term "set") borrows the *whole-parts relation* from space.[32] This explains why Hao Wang informs us that Kurt Gödel speaks of *sets* as being "quasi-spatial" – and then adds the remark that he is not sure whether Gödel would have said the "same thing of numbers" (Wang, 1988:202). The idea of wholeness or totality indeed has an original spatial meaning. Consequently, the notion of the power or cardinality of sets cannot be understood without acknowledging the interplay of *succession* and *at once* (*number* and *space*). For this reason, set theory ought to be appreciated as an arithmetical theory guided, directed and deepened by the core meaning of space (continuous extension).[33]

Weyl advances a similar view, and also uses the idea of *equinumerosity* (*Gleichzahligkeit*). He writes:

> Much discussion took place on whether cardinal number is the primary and ordinal number the secondary concept. ... This definition is not merely restricted to finite sets; connected to it Cantor developed his theory of infinite cardinal numbers within the context of his general set theory. Yet the possibility of mapping, which is intended in the criterion of *Gleichzaligkeit*, can only be assessed if the act of ordering takes one after the other, in an ordered temporal succession through which the elements of both sets themselves are ordered. ... Therefore it seems to me beyond doubt that ordinal number is primary. Modern mathematical foundational research, which is once again disrupted by dogmatic set theory, confirms this throughout.[34]

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31 "Bei der Analysis handelt es sich um die begriffliche Präzisierung geometrischer Vorstellungen" Bernays, 1976:VIII); "... daß die intuitionistische Vorstellung nicht jenen Charakter der Geschlossenheit besitzt, der zweifellos zur geometrischen Vorstellung des Kontinuums gehört. Und es ist auch dieser Charakter, der einer vollkommenen Arithmetisierung des Kontinuums entgegensteht" (Bernays, 1976:74).

32 In his discussion of the construction of ordinal numbers, Chihara also enters into an analysis of the issue of *impredicativity* (Poincaré, Russel, Weyl and Feferman) (Chihara, 1973:171 ff. and Chihara, 2004:19 note 13). Although this issue intimately coheres with the whole-parts relation, we will leave it aside in this context.

33 This view finds support in the conviction defended by Bernays, namely that the distinction between an 'arithmetical' and a 'geometrical' intuition should not be accounted for in terms of *space* and *time*, but by considering the difference between *discreteness* and *continuity*: "Es empfiehlt sich, die Unterscheidung von 'arithmetischer' und 'geometrischer' Anschauung nicht nach den Momenten des Räumlichen und Zeitlichen, sondern im Hinblick auf den Unterschied des Diskreten und Kontinuierlichen vorzunehmen" (Bernays, 1976:81)

34 "Es ist viel darüber gestritten worden, ob nicht umgekehrt die Kardinalzahl das Erste und die Ordinalzahl der sekundäre Begriff sei. ... Es ist diese Definition nicht einmal auf endliche Mengen

Although one may envisage a given mapping, such as $N \rightarrow N^2$, $a \rightarrow a^2$,[35] and then argue that it encompsses *all* of its elements at once, Weyl's orientation claims that an *assessment* (*prüfung*) of such a mapping has to fall back onto the succession of elements that is *pre-supposed* in every *at once representation*. *Checking* what is at stake entails that one looks at the mapping in its *successive instances* ($1 \rightarrow 1^2$, $2 \rightarrow 2^2$, $3 \rightarrow 3^2$, ...). In general one may therefore claim that the notion of *cardinality* can only be *assessed* by implicitly applying ordinality first. From the year 1922-1923, the article of Von Neumann sets out to establish an unequivocal (*eindeutig*) and concrete understanding of the Cantorean concept of an *ordinal number*. Although presented in terms of the language of naïve set theory, Von Neumann's analysis (in opposition to Cantor's approach) also maintains its validity when it is placed within the framework of the axiomatic foundation of set theory developed by Zermelo (by adding what eventually became known as Fraenkel's axiom schema of replacement). Von Neumann avoids the Cantorean term type (*Typus*) and assumes as the basis for his characterisation the statement: "Every ordinal number is the set of its preceding ordinal numbers."[36] Von Neuman starts with zero (which is equal to itself) and then proceeds by successively applying the recursive rule in the formation of sets with the preceding ordinal numbers as their elements.

0	=	0,
1	=	(0),
2	=	(0, (0)),
3	=	(0, (0), (0, (0))),
.	.	. . . . .
ω	=	(0, (0), (0, (0)), (0, (0), (0, (0))), ...),
ω + 1	=	(0, (0), (0, (0)), ... , (0, (0), (0, (0)) .....))

But he had to qualify the construction by pointing out that the statement: "Every ordinal number is the set of its preceding ordinal numbers" is not a proven statement about ordinal numbers. If transfinite induction already had been given a foundation, then this statement could have been a definition for it (Von Neumann, 1922/23:199). In other words, he does not pretend to provide a *foundation* for transfinite induction, but rather *assumes* that the concepts of a "well-ordered set" and 'Ähnlichkeit' are *given*.[37]

On the basis of the concept "natural number = *n*" Frege, by induction, establishes

beschränkt; die an sie sich knüpfende Theorie der unendlichen Kardinalzahlen hat G. Cantor im Rahmen seiner allgemeinen Mengenlehre entwickelt. Aber die Möglichkeit der Paarung, von der im Kriterium der Gleichzahligkeit die Rede ist, läßt sich nur prüfen, wenn die Zuordnungsakte einer nach dem andern, in geordneter zeitlicher Folge, vorgenommen und damit die Elemente beider Mengen selber geordnet werden. ... Daher scheint es mir unbestreitbar, daß die Ordinalzahl das Primäre ist. Die moderne mathematische Grundlagenforschung, welche die dogmatische Mengenlehre wieder zerstört hat, bestätigt dies durchaus" (Weyl, 1966:52-53).

35 Ben de la Rosa raised this possibility.

36 "Jede Ordnungszahl ist die Menge der ihr vorangehenden Ordnungszahlen" (Von Neumann, 1922-23: 199). This view became standard in axiomatic set theory. For example, Johan Heidema mentioned to me that in his work, "Axiomatic Set Theory" (1960), Patrick Suppes states: "If A is an ordinal then A = {B: B is an ordinal B A}," "each ordinal is just the set of smaller ordinals" (p.133, Theorem 11).

37 "Wir setzen aber natürlich die transfinite Induktion nicht als begründet voraus, wir nehmen vielmehr nur die Begriffe der »wohlgeordneten Menge« und der »Ähnlichkeit« als vorhanden an" (Von Neumann, 1922/23:200). Cantor calls simply ordered sets *similar* (*ähnlich*) when their elements can be correlated in such a way that the order-position (*Rangverhältnis*) of the correlated elements is the same in both sets: "ich nenne zwei einfach geordnete Vielheiten *ähnlich,* wenn sie eineindeutig so aufeinander

the "general theorem that the number of natural numbers = to any given natural number *n* is a successor of *n*; and from this the desired theorem that every natural number has a successor follows at once by existential quantification" (Dummett, 1995:124).

At this point Skolem is quite explicit about the impossibility to side-step the primitive meaning of the arithmetical order of succession – foremost evinced in the sequence of natural numbers and the nature of induction. He says that those oriented to set theory normally hold the view that the concept of a whole number must be defined and that complete induction must be proven. But according to him, it is clear that one cannot continue to define or provide a foundation *ad infinitum*, for sooner or later one arrives at what is not further to be defined or proven. The only option left is then to accept as basic what is immediately clear, natural and beyond doubt. This condition is met by the concept of a whole number and by inductive inferences, but it is definitely not met by set theoretical axioms such as those of Zermelo or something similar. If one wants to derive the former from the latter, then the set theoretical concepts must be simpler, and thinking with them must be more certain than that of complete induction – but this contradicts the real state of affairs.[38]

Intuitionistic mathematics observes in the basic arithmetical time-order of succession something mathematically *primitive*, but it also discerns in induction the guarantee that mathematics does not totally collapse into an enormous tautology.[39] Weyl actually defends the existence of synthetic judgments a priori in mathematics in the classical Kantian sense.[40] From his axiomatic formalistic orientation, Hilbert also affirms:

> Only when we analyze attentively do we realize that in presenting the laws of logic we already had to employ certain arithmetical basic concepts, for example the concept of a set and partially also the concept of number, particularly as cardinal number [*Anzahl*]. Here we end up in a vicious circle and in order to

bezogen werden können, daß das Rangverhältnis entsprechender Elemente bei beiden dasselbe ist" (Cantor, 1962:444).

38 "Die Mengentheoretiker sind gewöhnlich der Ansicht, dass der Begriff der ganzen Zahl definiert werden soll, und die vollständige Induktion bewiesen werden soll. Es ist aber klar, dass man nicht ins Unendliche definieren oder begründen kann; früher oder später kommt man zu dem nicht weiter Definierbaren bzw. Beweisbaren. Es ist dann nur darum zu tun, dass die ersten Anfangsgründe etwas unmittelbar Klares, Natürliches und Unzweifelhaftes sind. Diese Bedingung ist für den Begriff der ganzen Zahl und die Induktionsschlüsse erfüllt, aber entschieden nicht erfüllt für mengentheoretische Axiome der Zermelo'schen Art oder ähnliches; sollte man die Zurückführung der ersteren Begriffe auf die letzteren anerkennen, so müssten die mengentheoretischen Begriffe einfacher sein und das Denken mit ihnen unzweifelhafter als die vollständige Induktion, aber das läuft dem wirklichen Sachverhalt gänzlich zuwider" (Skolem, 1923:230).

39 "vom intuitionistischen Standpunkt [erscheint] die *vollständige Induktion* als dasjenige, was die Mathematik davor bewahrt, eine ungeheure Tautologie zu sein" (Weyl, 1966:86). Gödel holds that the existence of non-"tautological" relations between the concepts of mathematics "appears above all in the circumstance that for the primitive terms of mathematics, axioms must be assumed" (1995:320-321). The concept of a *set* depends on a basic axiom schema, but in the case of finitism, where the "general concept of a set is *not* admitted in mathematics proper ... induction must be assumed as an axiom" (Gödel, 1995:321).

40 "und prägt ihren Behauptungen einen synthetischen, nicht-analytischen Charakter auf" (Weyl, 1966: 86). Skolem also opposes logicists who believe that mathematical propositions are mere tautologies (see Skolem, 1929:12).

> avoid paradoxes it is necessary to come to a partially simultaneous development of the laws of logic and arithmetic.[41]

Although an *analysis* of the meaning of number requires an actual human functioning within the logical-analytical mode, this mode intimately coheres with the quantitative meaning of the numerical aspect. Quine correctly affirms that "[A]ny logic has to come to terms somehow with quantification, if it is not going to stop short" (Quine, 1970:88).

The logicistic thesis that number can be deduced from logic (Frege, Dedekind, Russell) turned out to be antinomous. The *Axiom of Infinity* (InfAx) actually *precedes* logic. Fraenkel *et al*. remark, "It seems, then, that the only really serious drawback in the Frege-Russell thesis is the doubtful status of InfAx, according to the interpretation intended by them" (Fraenkel *et al.*, 1973:186). Dedekind attempted to deduce the finite number from infinite classes and got stuck in similar difficulties (see Hilbert, 1925:167). Bernays refers to both Frege and Dedekind, who were indeed outstanding in the precision of their proofs and deliberations, but assumed as logical the apparently self-evident presupposition of the representation of a closed totality of all conceivable logical objects whatsoever.[42]

As an alternative to logicism, Hilbert maintained that mathematics possesses a content which is secured independently of all logic and that it can therefore never be based solely on logic.[43] Towards the end of his life, in 1924 or 1925, Frege confessed in his last attempt at providing a foundation for arithmetic, that he had to give up the opinion that arithmetic is a branch of logic and that accordingly, everything in arithmetic must be proven in a purely logical way.[44] The opposite position is found in Heyting, who asserts that logic is actually a part of mathematics: "This is the case for every logical theorem: it is but a mathematical theorem of extreme generality; that is to say, logic is a part of mathematics, and can by no means serve as a foundation for it" (Heyting, 1971:6).

At this point we may return to the example of Northrop, mentioned in the introductory paragraph, regarding the expedition of forty-three people needing food. Although it seems as if this example successfully makes the case for the primacy of cardinal numbers, the necessity of establishing a one-to-one correlation between the members of the expedition and the prepared plates of food by using *some or other* order of succession, in fact emphasises the priority of ordinality. Clearly, that ordinal numbers are indeed foundational is evident from the fact that identifying the cardinality of a set presupposes the recognition of some or other *order of succession*.

41 "Allein bei aufmerksamer Betrachtung werden wir gewahr, daß bei der hergebrachten Darstellung der Gesetze der Logik gewisse arithmetische Grundbegriffe, z. B. der Begriff der Menge, zum Teil auch der Begriff der Zahl, insbesondere als Anzahl, bereits zur Verwendung kommen. Wir geraten so in eine Zwickmühle, und zur Vermeidung von Paradoxien ist daher eine teilweise gleichzeitige Entwicklung der Gesetze der Logik und der Arithmetik erforderlich" (quoted by Bernays, 1970: 199).

42 "So waren Frege und Dedekind, deren Beweisführungen und Überlegungen sonst überall durch äußerste Präzision und Strenge ausgezeichnet sind, ganz unbedenklich in dem, was sie als vermeintlich selbstverständliche Voraussetzung dem Standpunkt der allgemeinen Logik zugrunde legten, nämlich in der Vorstellung von einer abgeschlossenen Gesamtheid aller überhaupt denkbaren logischen Objekte" (Bernays, 1976:47).

43 "daß die Mathematik über einen unabhängig von aller Logik gesicherten Inhalt verfügt und daher nie und nimmer allein durch Logik begründet werden kann" (Hilbert, 1925:171).

44 "Ich habe die Meinung aufgeben müssen, daß die Arithmetik ein Zweig der Logik sei und daß demgemäß in der Arithmetik alles rein logisch bewiesen werden müsse" (see Frege, 1983:298).

The set theoretical attempt to define an ordered pair

Before we conclude our argument in support of the foundational role of the quantitative time-order of succession (and its significance for acknowledging the foundational role of ordinality), we briefly have to look at the well-known definition of an ordered pair in set theory (and topology) in the light of our foregoing considerations.

In the set theory of Zermelo-Fraenkel (ZF) (with the membership relation as its only set-theoretical primitive symbol – see Fraenkel *et al*., 1973:22-23), one finds the notion of an ordered pair in their discussion of the *Axiom of Pairing*. The latter states that for any two elements *a* and *b*, there exists the set *y*, which contains just *a* and *b*. The initial understanding of an ordered pair states that

$$\text{for all } a, b, c, d, \text{ if } \langle a,b\rangle = \langle c,d\rangle \text{ then } a = c \text{ and } b = d.$$

Wiener and Kuratowski suggested that, within ZF, such a notion can be defined by

$$\langle a,b\rangle = \{\{a\}, \{a, b\}\}$$

This definition is 'burdened' by hidden assumptions – first of all the first-order predicate calculus alongside the primitive symbols that are taken from logic (such as connectives, quantifiers, variables and equality). Both an awareness of *multiplicity* and an intuition of *succession* are concealed in this underlying discipline. The mere reference to quantifiers and variables already exhibits the intuition of the *one* and the *many*. If there are more than one member in ZF (where the general form is "*x* is a member of *y*"), then both ordinality and cardinality is implicitly assumed (and successively explicated in the subsequent axioms of ZF – just consider the combination of the axioms of pairing, union and power-set).[45]

The definition of an ordered pair proposed by Wiener and Kuratowski simply begs the question, because although the sets $\{a\}$ and $\{a, b\}$ as such appear to exhibit *cardinality*, the number of members (respectively 1 and 2) hides an inevitable element of ordinality: there has to be a *first* set of the pair (with one member) and a *second* set of the pair (with two members). It is therefore not at all surprising that, in their initial formulation of the notion of an ordered pair – namely that for all *a*, *b*, *c*, *d*, if $\langle a,b\rangle =$ $\langle c,d\rangle$ then $a = c$ and $b = d$ – Fraenkel *et al*. had to insert the (circular!) qualifier "taken in that order": "The ordered pair $\langle a,b\rangle$ is an element which corresponds to *a* and *b* (taken in that order) such that ..." (Fraenkel *et al*., 1973:33).

Since the core meaning of number inevitably comes to expression in the primitive notions of *multiplicity* and (the arithmetical time-order of) *succession*, every attempt to define *order* (or something more specified like an *order pair*) will turn out to be *circular*.[46] Interrestingly the human infant appears to develop a sense of (numerical) order only at an age of 15 months, whereas at an earlier stage kids are already capable to observe a given multiplicity at once (called *subitizing*) (cf. Lakoff, & Núñez, 2000:18 ff.).

45 The axiom of power-set makes explicit the dependence of ZF on the original spatial whole-parts relation, for it postulates for any set *a* the existence of a set whose members are just all the subsets of *a* (see Fraenkel *et al*., 1973:35).

46 Of course this does not entail a "vicious circle" – as Johan Heidema also pointed out to me. In his discussion of the construction of ordinal numbers, Chihara also enters into an analysis of the issue of *impredicativity* (Poincaré, Russel, Weyl and Feferman) (Chihara, 1973:171 ff.; 2004:19 note 13). Although this issue intimately coheres with the whole-parts relation, we will leave it aside in this context.

Concluding remarks

Perhaps one may support Skolem in his critical questioning of attempts aimed at finding a set theoretical foundation for arithmetic, as if the latter did not already have a sufficient foundation in inductive conclusions and recursive definitions.[47] However, since intuitionistic mathematics and the (non-axiomatic) approach of Cantor (eventually) did have an eye for the primordial nature of succession and ordinality, mathematics may do well to recognise the quantitative time-order (*a la* Hamilton) as something primitive in our intuition of number. It is therefore not surprising that the notion of ordinality won the day in contemporary mathematics – in the sense of being more basic than that of cardinality. What is nonetheless particularly remarkable in this regard, is that when Cantor explains a *simple ordering* of a set, he does not hesitate to use terms reflecting the numerical time-order of succession. He writes:

> A set (*Vielheit*) is designated as 'simply ordered' when there exists between its elements an order of succession in such a way that, for every two elements, one is the earlier and the other the later, and that for every three elements, one is the earliest, the other the middle and the remaining one, according to its order-position (*Range*), the last among them.[48]

The implications of these considerations are twofold: (i) the notion of ordinality is more basic than cardinality, and (ii) the notion of cardinality, though exploring the whole-parts relation peculiar of spatial continuity,[49] at the same time (in a foundational sense) is unbreakably connected to the notion of ordinality, since the idea of a one-to-one correspondence entails a *successive* correlation. But since the insight that the notion of cardinality depends on the spatial features of *wholeness* and *at once* does not form a part of the arithmeticistic inclination to interpret sets in a purely arithmetical way, mathematicians may be reminded of the assessment of Paul Bernays:

> "It should be conceded that the classical foundation of the theory of real numbers by Cantor and Dedekind does not constitute a complete arithmetization. ... The arithmetizing monism in mathematics is an arbitrary thesis. The claim that the field of investigation of mathematics purely emerges from the representation of number is not at all shown. Much rather, it is presumably the case that concepts such as a continuous curve and an area, and in particular the concepts used in topology, are not reducible to notions of number (Zahlvorstellungen)."[50]

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47 "Es ist eigentümlich zu sehen, dass, weil die versuchte Begründung der Arithmetik auf die Mengenlehre nicht gut gelungen ist, wegen der logischen Schwierigkeiten der letzteren, so versucht man in sehr erkünstelten Weisen andere Begründungen zu finden, als ob nicht die Arithmetik schon an sich mit hilfe der Induktionsschlüsse und der rekurrenten Definitionen hinreichend begründet ware" (Skolem, 1923:231).

48 "Eine Vielheit heißt 'einfach geordnet', wenn zwischen ihren Elementen eine Rangordnung derart besteht, daß von je zweien ihrer Elemente eins das frühere, das andre das spätere ist, und daß von je dreien ihrer Elemente eines das früheste, ein anderes das mittlere und das übrig bleibende das dem Range nach letzte unter ihnen ist" (Cantor, 1962:444).

49 Bertrand Russell concedes: "The relation of whole and part is, it would seem, an indefinable and ultimate relation" (Russell, 1956:138).

50 Bernays, 1976:187-188: "Die hier gewonnenen Ergebnisse wird man auch dann würdigen, wenn man

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nicht der Meinung ist, daß die üblichen Methoden der klassischen Analysis durch andere ersetzt werden sollen. Zuzugeben ist, daß die klassische Begründung der Theorie der reellen Zahlen durch Cantor und Dedekind keine restlose Arithmetisierung bildet. Jedoch, es ist sehr zweifelhaft, ob eine restlose Arithmetisierung der Idee des Kontinuums voll gerecht werden kann. Die Idee des Kontinuums ist, jedenfalls ursprünglich, eine geometrische Idee. Der arithmeticsierende Monismus in der Mathematik ist eine willkürliche These. Daß die mathematische Gegenständlichkeit lediglich aus der Zahlenvorstellung erwächst, ist keineswegs erwiesen. Vielmehr Iassen sich vermutlich Begriffe wie diejenigen der stetigen Kurve und der Fläche, die ja insbesondere in der Topologie zur Entfaltung kommen, nicht auf die Zahlvorstellungen zurückführen".

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