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Defining mathematics

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Any definition of mathematics falls outside its field of investigation. When mathematics is set theory, the history of mathematics prior to the investing of set theory is eliminated. Arguing that the aspects of number and space delimit mathematics makes it possible to avoid both Platonism and constructivism in mathematics. Every philosophy of mathematics should be able to account for the nature and status of the infinite. That set theory is a spatially deepened theory of numbers cannot be accounted for by what Lakoff and Núñez call the Basic Metaphor of Infinity. Gödel's 1931 results point to an immediate, evident, intuitive insight.

Die definiëring van die wiskunde

Enige definisie van wiskunde beweeg buite die veld van ondersoek daarvan. As wiskunde versamelingsleer is, word the geskiedenis van die wiskunde voor die koms van versamelingsleer geëlimineer. Die argument dat die aspekte van getal en ruimte die wiskunde begrens, omseil beide die Platonisme en konstruktivisme. Elke filosofie van die wiskunde moet in staat wees om 'n verantwoording te gee van die aard en status van die oneindige. Dat die versamelingsteorie 'n ruimtelik-verdiepte teorie van getalle is, kan nie verantwoord word met behulp van wat Lakoff en Núñez aandui as "the Basic Metaphor of Infinity" nie. Gödel se 1931 resulate verwys na 'n onmiddellik-evidente insig.

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> Pure mathematics is the subject in which
> we do not know what we are talking about,
> or whether what we are saying is true.
> Bertrand Russell
> (*cf* Nagel & Newman 1971: 13)

In general special scientists appear to be convinced that the task of defining the discipline within which they are working should be assigned primarily to the specialists within the field. This entails that mathematicians should provide a (the) definition of mathematics, theologians one of theology, biologists of biology, and so on. This general point of view appears to apply to all the special sciences, which entails that in fact it precedes what is happening within these special sciences, for it is concerned with the general question: What is distinctive about any scientific or scholarly discipline?

This general question resides on a level of reflection transcending the differentiation of scientific disciplines. Its primary concern is to account for the difference between what is accepted as scholarly or scientific in nature and what is viewed as non-scientific in nature. Once this distinction has been articulated, a second one is in need of explanation, namely: How do we mutually distinguish between scientific disciplines (including mathematics and theology)? (*cf* Kuitert 1988: 19).

1. Theology as an example

The position of the discipline of theology is instructive in this regard. Within the Dutch South African legacy various theological faculties taught a scholarly discipline known as the "encyclopaedia of theology" to account for the nature of theology and for its various subdisciplines. As a rule this list included subdisciplines such as the dogmatological group, the bibliological group, the ecclesiological group and the missionary group. Whatever is included in this list, it never contains the said discipline: the encyclopaedia of theology. This practice therefore concedes that the discipline which has the task to account for the nature and subdisciplines of theology is itself not theological in nature. For any (open-ended) definition, such as "Theology is the discipline constituted by the following subdisciplines ....", does not (as definition) coincide with theology on the whole or with any of its parts (subdisciplines). Let us relate this situation to the problem of defining mathematics.

2. What is the nature of a definition of mathematics?

Suppose we say that mathematics is constituted by the disciplines of algebra and topology (for example, by the most general disciplines of number and space).[1] We may substitute this definition with any other one, such as saying that mathematics is the science of formal structures (Körner 1968: 72, Meschkowski 1972b: 356) or that it is concerned with idealised structures (Bernays 1976: 176), or that it is the science of the infinite (Weyl 1966: 89) – each time the given definition will not be a part of mathematics. For example, the statement "mathematics is the science of formal structures/idealised structures/the infinite, and so on" is not a theorem, proof or derivation found in algebra, topology, set theory or any other part of mathematics.

3. Is mathematics set theory?

Although Cantor's set theory experienced a setback in the discovery of the so-called antinomies, particularly (and independent of each other) those made known by Russell and Zermelo (in 1900 and 1901) in connection with the set of all sets that do not have themselves as an element, the majority of mathematicians currently still appreciate it as the foundation of their discipline.

Throughout the twentieth century many mathematicians went a step further in claiming that ultimately mathematics is set

1 The French mathematicians known as Bourbaki worked on the following subdisciplines of mathematics: set theory (*théorie des ensembles*); algebra (*algèbre*); topology (*topologie générale*); functions of one real variable (*fonctions d'une variable réelle*); topological vector spaces (*espaces vectoriels topologiques*); integration (*intégration*); commutative algebra (*algèbre commutative*); Lie theory (*groupes et algèbres de Lie*), and spectral theory (*théories spectrales*).

theory (*cf* Maddy 1997: 36, Hersh 1997: 27). Maddy (1997: 22) remarks:

> The view of set theory as a foundation for mathematics emerged early in the thinking of the originators of the theory and is now a pillar of contemporary orthodoxy. As such, it is enshrined in the opening pages of most recent textbooks.

Yourgrau (2005: 72) also remarked: "Even today, the axioms of Zermelo-Fraenkel's set theory are the most widely used and accepted in the field". Most of the time the general and concise statement simply is: "mathematics is (axiomatic) set theory".

4. The philosophy of mathematics

As far as the question "What is theology?" is concerned, Popma realised that, although it is not a special scientific question, it still is a scientific issue. But because it appears at the foundation of every special science, it is clearly a general philosophical question (*cf* Popma 1946: 13). The implication is that the above-mentioned definitions of mathematics, including the view that mathematics is set theory, also belong to the philosophical foundations of mathematics and not to the inner workings of the discipline of mathematics itself. Hersh emphasises that the majority of "mathematicians hold contradictory views on the nature of their work". He points out that truth and meaning are non-peripheral issues and if ignored they may leave mathematicians "captive to unexamined philosophical preconceptions" (Hersh 1997: 40).

Hersh mentions the unfortunate introduction of set theory and its axiomatics into primary and secondary schools in the 1960s as an example of a philosophical presupposition:

> This wasn't an inexplicable aberration. It was a predictable consequence of a philosophical doctrine: Mathematics is axiomatic systems expressed in set theoretic language (Hersh 1997: 41).

Hersch advanced his own alternative philosophy of mathematics in opposition to "Platonism and formalism and neo-Fregianism". He believes that mathematics "comes first, then philosophizing about it" (Hersh 1997: xi; a related view is found in Shapiro 2000:

7). However, the view expressed in his last-mentioned statement generates basic questions regarding the inevitable assumptions needed by mathematicians in order to do mathematics.

In addition, although a definition of mathematics does not belong to the discipline itself but to its philosophical foundations, it is equally true that every philosophy of mathematics is always influenced by the developments taking place within this discipline. However, this concerns a different problem than the one investigated in this article. Whereas the question "What is mathematics?" leads us to the philosophical foundations of mathematics, considering the influence of the discipline of mathematics on the philosophy of mathematics takes us back to the content of mathematics itself. In the reflection of the current article the main focus is on talking about mathematics and not on doing mathematics. Even if the influence of the discipline of mathematics upon the philosophy of mathematics is stronger than that of the philosophy of mathematics upon the discipline itself, does it not disqualify the perspective defended below, namely that the question "What is mathematics?" does not belong to the field of investigation of mathematics.

5. An intrinsic philosophical question: what is distinctive about the scholarly enterprise?

If philosophy comes after mathematics, then no intrinsic philosophical assumptions operate in the practice of mathematics. Let us assess this claim by first comparing the situation in mathematics with what is found within the discipline of biology, against the background of what is regarded as the distinctive feature of scholarly activities.

5.1 Ambiguities

Philosophers of science contemplated features such as systematics, verification/falsification, methodology, the relation between a knowing subject and the object of knowledge, and the nature of abstraction, as candidates possibly capable of explaining what is unique about the scholarly enterprise. Yet, closer investigation has shown that all these features are found both within non-scientific and scientific contexts. There are non-scientific situations where one has to proceed in a systematic way, obtain certainty about things (the ingredients of a cake, the evidence in a court case), using methods, and relate to "objects". Not even the subject-object relation is exclusive or decisive, because every concrete "object" displays functions that allow access for different academic disciplines (the same business firm can be investigated scientifically by business economics, industrial sociology and industrial psychology). The same ambiguity inheres in the property of abstraction. There are scientific and non-scientific forms of abstraction, apart from the fact that abstraction may be geared towards concrete entities (things) or towards modes or aspects of such concretely existing things. Frege, for example, believed that abstraction exclusively holds for higher levels of entitary abstractions. His aim was to show that the concept of number cannot be derived by means of abstraction. His famous example is the moon. Through abstraction one can only arrive at more general concepts such as "attendant of the earth", "attendant of a planet", "celestial body without its own light", "celestial body", "body", "object" (*Gegenstand*) – and nowhere in this series will the number '1' occur (Frege 1884: 57). Frege also uses the example of a white cat and a black cat in order to highlight the shortcomings of 'abstraction':

> The concept 'cat', that has been obtained through abstraction does indeed contain no particulars, but precisely for that reason it is only one concept (Frege 1884: 45-6; translation by Dummett 1995: 84).[2]

5.2 Scientific thinking: the role of an abacus

Therefore it must be clear that abstraction in the context of entities cannot produce the kind of functional concepts within a scientific universe of discourse (such as the concept of number).

2 Tait (2005: 241) holds that Frege tends to confuse the following two questions: "What are the things to which number applies? What are numbers?"

Those of us who had to use an abacus in order to be taught how to do the most basic arithmetical operations, such as addition and subtraction, will understand this. While learning to calculate with the help of the abacus, we commenced by involving different aspects of reality at once. We take into account the colour, the movement, the shape and the quantity of blocks on the abacus, that is, we initially include the many-sidedness of these blocks by leaving their physical, kinematical, spatial and numerical properties intact. But as soon as we start to ignore the colour, movement and shape of these blocks and concentrate our attention solely on the quantitative side, we had to elevate the numerical aspect by simultaneously ignoring the non-numerical aspects (the spatial, the kinematic, the physical and the other aspects not yet mentioned).[3] One may say that the moon has subject functions within the four mentioned aspects but object functions in all the post-physical aspects.

5.3 Different kinds of abstraction

Yet, Frege neglected a different kind of abstraction; abstraction directed towards the identification of specific aspects of reality. This can be achieved only when (i) there is more than one aspect and (ii) a specific aspect is distinguished from all the other aspects. Since analysis (identification and distinguishing) and abstraction (lifting out while disregarding) are in fact synonymous, one can say that modal analysis or modal abstraction is the distinctive feature of scientific endeavours.

Regarding point (i): Once it is realised that diverse special sciences are demarcated by distinct modal (aspectual) points of entry to reality (delimiting their respective fields of investigation), it must be clear that more than one aspect is required in order to identify one specific mode (modality). Point (ii) entails an obvious implication, namely that the answer to the question regarding the nature of any special science always exceeds the confines of

3 Bear in mind that the abacus is a cultural artifact (its formative aspect), that it has a name (its function within the sign mode), that it belongs to someone (the jural aspect evinced in the accompanying property right), and so on.

the discipline concerned. An example related to what is currently known as plant science and animal science may elucidate this point. Suppose we say that "plant science is a study of plants". This statement does not say anything specific about plants, for as such it is solely focused on the discipline investigating plants. If we step back and ask the question: "what is a plant?" the first reaction may be to say that only botany as an academic discipline can tell us what a plant really is. But is this the case? Suppose there has never been a botanist and for the first time someone commences with a scholarly investigation of the nature of plants. How does that person know what a plant is if there is no textbook on plant science? Are there any guarantees that our first "botanist" indeed investigates plants? If there is no plant scholar who can tell her what plant-ness is all about, what would prevent our first "botanist" from investigating material things or animals while being under the impression that they are plants?

However, every human being does have a pre-scientific acquaintance with the given diversity in nature, that is, with the similarities and differences between things, plants, animals and human beings. The concepts formed of these entities could be named at will, without eliminating the inherent natures to which they refer. This argument therefore presupposes a clear distinction between concept and word as well as an implicit appeal to what is known, within philosophical epistemology, as the problem of evidence. Stegmüller (1969: 194) points out:

> Some form of an absolute knowledge must exist; without it we would not have been able to begin; We must already 'possess' absolute evidence, that is, we must already believe in it.[4]

After Hilbert's death in 1943, his student, Hermann Weyl (1970: 269), who switched to an intuitionistic orientation, wrote:

4 "Irgendein absolutes Wissen muß es geben; ohne dieses könnten wir überhaupt nicht beginnen"; "Absolute Evidenz müssen wir schon 'haben,' d.h. wir müssen an sie bereits glauben, ..."

> It must have been hard on Hilbert, the axiomatist, to acknowledge that the insight of consistency is rather to be attained by intuitive reasoning which is based on evidence and not on axioms.

Without the obvious differences between things, plants, animals and human beings no scholarly reflection on them would be possible. These considerations support our conclusion that without a prior knowledge of the nature of plants, not even a (first) botanist will be able to study plants. This prior knowledge has no other basis than the above-mentioned everyday, pre-scientific awareness of the diversity in our experiential world. In other words, ultimately not even the discipline of plant science can operate by negating our non-scientific knowledge about the world.

Various philosophical trends, particularly over the past 100 years, emphasised the importance of acknowledging the embeddedness of ordinary human activities within the (intersubjective) human life-world (German: *Lebenswelt*). In everyday life, our logical thinking is embraced by an awareness of a more-than-logical diversity, and it is merged within this diversity in multiple contexts. This pre-scientific awareness is not something that ought to be eliminated or denied by scholarly thinking, since it forms the unavoidable basis and starting point of scientific reflection. Likewise, without an antecedent experiential knowledge of multiplicity and succession, mathematics cannot tell us what numbers are all about.

6. Human experience of the world

Mathematics is also dependent upon a prior (for example, pre-scientific) acquaintance with the world. Human beings are collectively and individually merged into the diverse aspects of the universe. Those involved in cognitive science, cognitive psychology, cognitive linguistics and what became known as the embodied mind are attentive to the embodied experiences human beings have of the various dimensions of our world (cf Lakoff & Johnson 1999, Lakoff & Núñez 2000).

It appears that the recent developments within cognitive science are compatible with the notion that concrete, lived-through experiences are lying at the basis of most of our thinking and that this foundation primarily manifests itself in the universal presence of conceptual metaphor. But we have to take one step further back, because what is implicitly presupposed in all so-called conceptual metaphors are multiple ontic modes of being, in other words, truly existing aspects of reality within which human beings are functioning in a concrete and many-sided way.

7. Platonism

The Platonic tradition within philosophy and mathematics holds the view that there is a so-called "mind-independent reality" out there, and that mathematicians simply discover a pre-existing mathematical world. Traditionally this stance is also designated as realistic. It concerned what became known as universals and these universals were supposed to have a threefold existence: *ante rem* (in God's mind – the legacy of Plato), *in re* (within concretely existing things as their universal substantial forms – the legacy of Aristotle) and finally *post rem* (as universal concepts or words within the human mind).

In 1934 Paul Bernays, colleague of the foremost mathematician of the twentieth century, David Hilbert, presented a paper on Platonism in mathematics. He pointed out that Platonist conceptions not only extend far beyond the theory of real numbers, for they have proved to be very fertile in "modern theories of algebra and topology". His brief overview of the application of Platonism in mathematics shows, according to him, that this "application is so widespread, that it is not an exaggeration when it is said that Platonism currently reigns in mathematics" (Bernays 1976: 65).[5]

5 "Diese Anwendung ist eine so übliche, daß es keine Übertreibung ist, wenn man sagt, der Platonismus sei heute herrschend in der Mathematik."

8. A coherence or correspondence theory of scientific truth?

It is remarkable, however, that since the Renaissance modern nominalism prevailed within the development of modern philosophy. It denied any universality outside the human mind and therefore rejected *universalia ante rem* and *universalia in re*. Universality is found only within the human mind. For example, Descartes holds that number and all universals are mere modes of thought (*Principles of Philosophy*, Part I, LVII). Hobbes advocated a similar conviction: "Truth does not inhere in the things, but is attached to the names and their comparison as they are employed in statements" (*cf* Cassirer 1971: 56).[6]

This controversy gave rise to what is currently still known as the opposition between the correspondence and the coherence theories of truth. The classical realistic tradition views truth to be based upon an *adequatio intellectus et rei* (the correspondence between thought and being), while nominalism is concerned with the compatibility (coherence) of concepts.

One cannot understand this controversy without giving an account of the logical-analytical subject-object relation. Objectification is always the act of a subject. The perception of a stone opens up its perceivability, makes patent its latent sensory object-function. Appreciating the beauty of a sunset objectifies its latent aesthetic object function within this aspect. The sunset is a physical event (a subject within the physical aspect of reality). It cannot objectify itself in the aesthetic mode. Only an aesthetic subject (such as an appreciative human observer) can objectify the sunset within the aesthetic aspect. The same applies to the logical-analytical subject-object relation. The human logical-analytical ability to discern, to identify and to distinguish can result in logical objectification when whatever is identifiable and distinguishable is actively identified and distinguished.

6 "Die Wahrheit haftet nicht an den Sachen, sondern an den Namen und an der Vergleichung der Namen, die wir im Satze vollziehen: veritas in dicto, non in re consistit" (cf *De Corpore*, Part I, chapter 3, paragraphs 7 & 8).

9. The logical-analytical subject-object relation

The opposition between a correspondence theory of truth and a coherence theory of truth unilaterally emphasises either the factual object side or factual subject side of the logical analytical aspect. In its universal scope (its modal universality), this aspect embraces whatever there is and therefore underlies the ability we have to logically objectify whatever is identifiable and distinguishable. Although the conceptual framework within which knowledge is embedded co-determines our knowledge acquisition (the main focus of Kant's epistemology), it is always at once related to what is logically objectified (compare the difference between the connotation and denotation of a word or sentence occurring within the lingual subject-object relation).

To phrase it differently: When someone (as a logical-analytical subject) constructs a theory of numbers or a theory of sets, then we always have to account both for the subjective construction and for the correlated objectifiable content. If it was not the case that the entire universe had either a subject-function or an object-function within every aspect, human beings would have been opposed to things with which nothing is shared. Stafleu (1999: 100) therefore correctly holds that "human thought is subject to the same kind of laws as the creation as a whole; this is even a condition for the achievement of knowledge".

10. Constructivism

When the theme of construction obtains the upper hand in our reflections, we may think that whatever happens within mathematics is nothing but (arbitrary) human constructions. Paul Lorenzen advanced his constructive mathematics in the 1960s and before him modern intuitionism (Brouwer and his followers) also developed the notion that mathematical existence coincides with constructibility. Even those who adhere to the approach of axiomatic formalism may view such axioms merely as subjective thought-constructions of human beings (mathematicians).

11. Ontic conditions: modal functions

Yet, we have to distinguish not only between number and space (discreteness and continuity),[7] but also between these two and the logical-analytical aspect. In addition, the key terms involved in rational conceptual understanding are themselves not open to (rational) conceptual definition, for they are, as Cassirer puts it, *Urfunktionen* (original functions). Cassirer holds that there are "original functions that are not in need of genuine derivation". He also realised that there are relations between different functions (aspects) of reality, for he speaks of "original functions" and their interconnections. He refers, in particular, to the similarity and difference between a logical identity and diversity and a numerical unity and difference.[8]

These ontic conditions not only make possible our concept of numbers but also explain why someone like Bernays (1976: 45) rejects the notion that an axiomatic system in its entirety is an arbitrary construction:

> One cannot justifiably object to this axiomatic procedure with the accusation that it is arbitrary since in the case of the foundations of systematic arithmetic we are not concerned with an axiom system configured at will for the need of it, but with a systematic

7 A few references regarding the centrality of the issue of discreteness and continuity for the foundation of mathematics are given in this instance. Fraenkel *et al* (1973: 211) hold that "[B]ridging the gap between the domains of discreteness and of continuity, [...] is a central, presumably even the central problem of the foundation of mathematics". Brouwer's (1964: 69) "basal intuition" embraced both the elements of discreteness and continuity. When Paul Bernays considers the distinction between our arithmetical and geometrical intuition he rejects the widespread view that it concerns time and space. According to him, the proper distinction is that between the "discrete" and the "continuous" ["Es empfiehlt sich, die Unterscheidung von 'arithmetischer' und 'geometrischer' Anschauung nicht nach den Momenten des Räumlichen und Zeitlichen, sondern im Hinblick auf den Unterschied des Diskreten und Kontinuierlichen vorzunehmen"] (Bernays 1976: 81).

8 "In der Tat ist nicht einzusehen, warum man lediglich logische Identität und Verschiedenheit, die als notwendige Momente in den Mengenbegriff eingehen, als solche Urfunktionen gelten lassen und nicht auch die numerische Einheit und den numersichen Unterschied von Anfang an in diesen Kreis aufnehmen will" (Cassirer 1957: 73-4).

> extrapolation of elementary number theory conforming to the nature of the matter (*naturgemäß*).[9]

The "nature of the matter" contains an implicit reference to the ontic status of the "multiplicity aspect" of reality and it presupposes an awareness of the difference between the various (modal, functional) aspects of reality and the concrete dimension of entities and events functioning within these aspects.

12. Mathematics is not entirely an arbitrary construction

It is noteworthy that none other than Kurt Gödel also opposed the notion of a complete, arbitrary, construction in mathematics. In his discussion of Russell's mathematical logic he holds "that logic and mathematics (just as physics) are built up on axioms with a real content which cannot be 'explained away' " (*cf* Gödel 1964: 224; for an analysis of Gödel's realism, *cf* Shapiro 2000a: 202-11).

At this point, where we have referred to prominent mathematicians who objected to the notion of making mathematics a completely arbitrary affair by acknowledging an ontic point of departure, we may pay attention to the important implication of this stance for an understanding of what mathematics really is. If mathematics is merely what mathematicians created at a specific time in history, then it becomes impossible to speak about the history of mathematics.

One implication of this test is that a discipline such as mathematics may learn from its mistakes and therefore develop in a self-correcting way.[10] It is interesting to note that the history of mathematics demonstrates two tendencies, for alongside self-

9 "Gegen diese axiomatische Vorgehen besteht auch nicht etwa der Vorwurf der Willkürlichkeit zu Recht, denn wir haben es bei den Grundlagen der systematische Arithmetik nicht mit einem beliebigen, nach Bedarf zusammengestellten Axiomensystem zu tun, sondern mit einer naturgemäßen systematischen Extrapolation der Elementare Zahlenlehre".

10 One reviewer raised this issue.

correction it sometimes appears to return (though in a more sophisticated way) to basic philosophical orientations belonging to a distant past. For example, Greek mathematics started with the Pythagorean emphasis everything is number, then switched to a spatial perspective which dominated the scene until early modernity when Descartes started to revert to an arithmeticistic perspective. The latter was carried through in the late nineteenth century by Weierstrass, Dedekind and Cantor. However, although Frege initially proceeded from the same arithmeticistic ideal, the discovery of Russell's antinomy in 1900[11] uprooted the assumptions of his two-volume work: *Grundgesetze der Arithmetik* (1893 and 1903). The effect was that, by the end of his life, he once again reverted to a spatial orientation (*cf* Frege 1979: 277).[12]

In this instance, we may also refer to the more recent views of René Thom and others who present themselves as "mathematicians of the continuum" who hold that "the continuum precedes ontologically the discrete", for the latter is merely an "accident coming out of the continuum background", "a broken line" (*cf* Longo 2001: 6, 19 & 20).

13. The test of the history of mathematics in defining mathematics

The crucial question therefore concerns the history of mathematics. If mathematics is set theory, then the history of mathematics is merely the history of set theory. But every mathematician knows that set theory (and modern mathematics) emerged a mere 137

11 Bertrand Russell and Ernst Zermelo independently discovered the intrinsically problematic nature of the notion of a set and its elements (*cf* Husserl 1979: xxii, 399).

12 "So an a priori mode of cognition must be involved here. But this cognition does not have to flow from purely logical principles, as I originally assumed. There is the further possibility that it has a geometrical source. [...] The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis – a geometrical one in fact – so that mathematics in its entirety is really geometry" (Frege 1979: 277).

years ago when Georg Cantor gave his first proof of the non-denumerability of the real numbers. Therefore, on this view, mathematics is a very young discipline. Without "anchoring" mathematics in a given reality, we seem to be doomed to be unable to account for the history of this discipline. In positioning himself *vis-à-vis* Platonism Hersh (1997: 42) raises the question: "[T]o what objects or features of the world do such statements refer?"

Hersh understands the importance of this insight. Human theoretical thinking may disclose and deepen the meaning of number (and space) in numerous ways, but this cannot be accomplished in a purely arbitrary manner. As far as mathematical objects and the infinite are concerned, he holds: "Though they are our inventions, their properties are not arbitrary" (Hersh 1997: 24). He is sharply critical of the reductionist view that mathematics is set theory because one cannot say that those mathematicians who lived long before the invention of set theory in fact thought in terms of set theory. His categorical statement reads: "This claim obscures history, and obscures the present, which is rooted in history" (Hersh 1997: 27), adding the important remark: "An adequate philosophy of mathematics must be compatible with the history of mathematics. It should be capable of shedding light on that history" (Hersch 1997: 27).

14. The question is not: 'Who defines mathematics?' but 'What is the nature of a definition of mathematics?'

Since the task to define mathematics falls within the domain of philosophy, it should be understandable that related (classical) philosophical problems implicitly play a role in our attempts to understand what mathematics is. Of course, mathematicians may feel uncomfortable with such an assessment, because they may still want to defend their turf, by claiming that the only person who can give a definition of mathematics is a mathematician. Yet this objection shows that those who support this move still bypassed the real issue. The question is not "Who defines mathematics?" but "What is the nature of a definition of mathematics (whether or not given by a mathematician)?"

Whoever attempts to define mathematics has to take a step back, a step into the philosophy of mathematics, because it is only from this vantage point that one (also as a mathematician) can speak about mathematics. In other words, even if a mathematician takes on the task to define mathematics, the resulting definition does not belong to the discipline of mathematics but rather to the (foundational) philosophy of mathematics. The issue is thus not "Who gives the definition?" but "What is the nature of the definition?"

15. Another defining test for mathematics: does the infinite exist?

Among the requirements stipulated by Hersh (1997: 24) as a test for every philosophy of mathematics one finds the questions: "Does the infinite exist? How?"

This statement reminds us of the words of Hermann Weyl alluded to above, namely that mathematics is the science of the infinite. His formulation reads as follows: "If one desires to give a brief characterization touching the vital core of mathematics, then one can pretty well say: it is the science of the infinite."[13] David Hilbert (1925: 163 & 1964: 136) anticipated this view with his statement:

> From time immemorial the infinite has stirred men's emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully. Yet no other concept needs clarification more than it does.

Let us, in conclusion, briefly reflect on the role of infinity in defining mathematics.

13 "Will man zum Schluß ein kurzes Schlagwort, welches den lebendigen Mittelpunkt der Mathematik trifft, so darf man wohl sagen: sie ist die Wissenschaft vom Unendlichen" (Weyl 1966: 89).

Mathematicians are all very well acquainted with the basic given of a succession of numbers, normally found in simple acts of counting. The quantitative order of succession guarantees the uniqueness of every successive number and underlies the most basic awareness of infinity, literally without an end, endlessly. The attempt to define number in terms of a combination of "pure ones" was severely and effectively criticised by Frege, because "pure ones" will always collapse into the general concept of oneness – which is incapable of a plurality (*cf* Frege 1934: § 45).[14] This order of succession underlies the principle of induction and, according to Weyl, it provides the safeguard preventing mathematics from collapsing into one enormous tautology. Weyl (1966: 85-6) makes this remark when he deals with the essence of mathematical knowledge (*Über das Wesen der mathematischen Erkenntnis*).

16. A classical distinction

The kind of infinity which Weyl has in mind is the only kind acknowledged by intuitionistic mathematics, traditionally known as the potential infinite. The intuition of multiplicity is made possible by the unique quantitative meaning of the numerical aspect – first accounted for in the introduction of the natural numbers and in the fact that succession is also inherent within our understanding of natural numbers.[15] But the additional step present in Cantor's set theory appears to be less certain than what can be achieved on the basis of the integers – to which already Kronecker wanted to reduce all of mathematics. Skolem (1979: 70) summarised his assessment in 1922 as follows:

> Those engaged in doing set theory are normally convinced that the concept of an integer ought to be defined and that complete induction must be proved. Yet it is clear that one cannot define or provide an endless foundation; sooner or later one encounters

14 This demonstrates the difference between arithmetical addition and logical addition.

15 Dedekind (1969: paragraph 59, 80) was the first to call the conclusion from n to $n + 1$ complete induction ("vollständige Induktion").

> what is indefinable or non-provable. Then the only option is to ensure that the first starting points are immediately clear, natural and beyond doubt. The concept of an integer and the inferences by induction meet this condition, but it is definitely not met by the set theoretic axioms such as those of Zermelo or similar ones. If one wishes to derive the former concepts from the latter, then the set theoretic concepts ought to be simpler and employing them then ought to be more certain than working with complete induction – but this contradicts the real state of affairs totally.

Likewise, Mostowski holds that the "raison d'être" for the number concept is found in our natural and real experience (Meschkowski 1972b: 344).

17. Complications – the radical difference between mathematicians

While all schools of thought within mathematics have peace with the potential infinite, the picture drastically changes as soon as the actual infinite emerges. The most authoritative attack on the actual infinite, often also designated as the completed infinite, is found in a letter from Gauss to Schumacher (12 July 1831):

> So I protest against the employment of an infinite magnitude as something completed, which, within mathematics, is never allowed (Meschkowski 1972a: 31).[16]

However, rendering the distinction between the potential infinite and the actual infinite in terms of the incompleted and completed infinite is both misleading and counter-intuitive. The standard illustration of the potential infinite uses the counting numbers in their natural succession. Initially I adhered to the practice of referring to such a succession as the potential infinite or as the incompleted infinite (*cf* Strauss 1983). More than a decade later it was clear to me that the most suitable characterisation of the two kinds of infinity is given in the phrases "successive infinite" and "at once infinite" because this distinction embraces two different aspects of reality (*cf* Strauss 1996: 235). It is

16 "So protestiere ich gegen den Gebrauch einer unendlichen Größe als einer vollendeten, welches in der Mathematik niemals erlaubt ist."

interesting to note that these phrases are also found among those theologians who speculated about the infinity of God during the transition from the late Middle Ages to the modern era (early fourteenth century). The Latin phrases are *infinitum successivum* and *infinitum simultaneum* (*cf* Maier 1964: 77-9). The most recent explanation of the way in which these two expressions presuppose both the uniqueness and mutual coherence of number and space is found in Strauss (2009: 235-42) where §5.20.2 is dedicated to "Mathematics and the nature of infinity".

18. Set theory as a spatially disclosed mathematical theory

The outcome of this analysis is the view that set theory should be regarded as a spatially disclosed number theory. In set theory the meaning of number is revealed under the guidance of the regulative hypothesis of the "at once infinite". This deepening of meaning enables the notion of infinite totalities, clearly a reference (anticipation) from number to the original spatial whole-parts relation. Hersh (1977: xi) is mistaken in his belief that mathematics first had to be there before the philosophy of mathematics can reflect on it. He did not realise that one or another notion concerning the uniqueness and coherence of number and space (implicitly or explicitly) is presupposed in the doing of mathematics. How else can we explain that a seemingly exact mathematical proof, such as Cantor's diagonal proof for the non-denumerability of the real numbers, yields opposite conclusions dependent on the kind of infinity that is assumed, the "successive infinite" or the "at once infinite".[17] If modal abstraction is acknowledged as the distinctive feature of scholarly activities, then accounting for the "at once infinite" is philosophical in nature because it has to consider the relationship between more than one aspect, namely the numerical and spatial aspects. The only way to justify the "at once infinite" is to realise that it is an anticipation from the numerical aspect to the spatial

17 In the first case, no non-denumerability follows, *cf* Strauss 2011.

order of simultaneity. This anticipation forms the foundation of the notion of infinite totalities. This approach therefore proceeds from acknowledging the ontic basis of our numerical and spatial intuitions, earlier, as suggested by Paul Bernays, related to the notions of discreteness and continuity.

19. Conceptual metaphor and the Basic Metaphor of Infinity (BMI)

In their theory of conceptual metaphor, Lakoff & Núñez (2000: 324) believe that continuity and discreteness are opposites. Within different aspects we do find opposites, but different aspects are not opposing each other. For example, within the numerical mode we find opposites such as many and few, within space we meet opposites such as big and small, in the kinematic mode we meet slow and fast, in the physical strong and weak (or: heavy and light), in the biotic healthy and sick, and so on. But unique (and irreducible) aspects are not opposites. The option not pursued by Lakoff & Núñez is to realise that discreteness and continuity belong to mutually cohering but distinct ontic functions or aspects of reality.

In their discussion of what they call embodied infinity Lakoff & Núñez argue from the perspective of our concrete bodily experience of acts and events. They point out that an action such as breathing is inherently iterative (Lakoff & Núñez 2000: 156). Since such actions are conceptualised as not completed, this imperfective aspect is regarded as "the fundamental source of the concept of infinity", that is, of the literal concept of infinity outside mathematics (Lakoff & Núñez 2000: 156). They proceed with their argument by alluding to continuous action, conflated into a conceptualisation of repeated actions (Lakoff & Núñez 2000: 157). Literally speaking, they hold that there is "no such thing as the result of an endless process" (Lakoff & Núñez 2000: 158). They believe that all cases of actual infinity are "special cases of a single general conceptual metaphor in which processes that go on indefinitely are conceptualized as having an end and an ultimate result". This is what they call the "Basic Metaphor of Infinity" (BMI).

According to Lakoff & Núñez (2000: 160), the BMI "is a product of human cognition, not a fact about the external world". The symbol ∞ is regarded as the largest integer (used for enumeration and not for calculation), but the "BMI itself has no numbers" (Lakoff & Núñez 2000: 166).

20. Critical appraisal

Let us examine these distinctions. Using the term "largest" is quite significant, because what is at stake is a quantitative (numerical) property. If we want to restrict ourselves to the domain of number, one would rather have expected terms related to "more" and "less", that is to say to terms referring to the "many-est" (largest) number with, as its numerical opposite, the "few-est" (smallest) number. But we still did not escape from terms that are derived from the modal (aspectual) meaning of space, because we used the expression "the domain of number". The fact that mathematicians still speak of the infinitely large and the infinitely small demonstrates that an analysis of the meaning of number cannot avoid employing terms derived from non-numerical aspects.[18]

Do we have to conclude, flowing from the use of the opposition between large and small or from the use of the term domain, that the aspect of space itself is also merely a product

18 Although the quantitative meaning of number is *primitive* and *indefinable*, every attempt to analyse its meaning inevitably employs the use of non-numerical terms. For this reason, the intuitionistic approach of Dummett, while rejecting the idea of infinite totalities, still had to use the spatial term domain in the expression "infinite domain" (*cf* Dummett 1978: 22, 24, 57).The same applies to terms derived, for example, from the kinematic and physical aspects. No mathematician or expert in the field of mathematical logic normally realises that the "second nature" terms, constants and variables, are actually made possible by the unique (and irreducibile) meanings of the kinematic and physical aspects – evinced in a uniform (constant) motion, which forms the basis of establishing change (variation).

of human cognition? The same question applies to the iterative aspect of "continuous" human actions. What makes possible any concrete iterative process? If iteration is but "a product of human cognition", how does one explain any succession in nature which occurs independently of human cognition, such as the succession of day and night?[19] Similarly, is the experience we have of sizes (small and big) or areas (domains) solely a product of human cognition?

What about the cognising subject? If it can be shown that human beings are capable of forming the concept of a square or a circle, does that mean that human cognition "created" the property of squareness or that of circularity?[20]

21. Uniqueness and coherence

We should rather argue for a recognition of number and space as two unique ontic modes of reality making possible the development of mathematics as a special science. Once this is done, then it is possible to allow for something given and for the theoretical disclosure of the meaning of what is given.

When, under the guidance of our theoretical (that is, modally abstracting) insight into the meaning of the spatial order of simultaneity, the original modal meaning of the numerical time order of succession is disclosed, we encounter the regulatively deepened anticipatory notion of actual infinity or the at once infinite. Any succession of numbers may then, directed in an anticipatory way towards the spatial order of simultaneity, be

19 Kant already noticed the difference between causality (cause and effect) and succession. Although the day is succeeded by the night and *vice versa*, one cannot say that the day is the cause of the night, or that the night is the cause of the day.

20 Our critical questioning of the products of human cognition can be extended to include an account of the status of the logical principles of identity and (non-)contradiction. If there are no such given logical principles both the identity judgment "A is A" and the recognition of an illogical concept (such as a "square" circle) would be impossible.

considered as if its infinite number of elements is present as a whole (totality) all at once.

Contrary to both the Platonist and the Constructivist, we thus acknowledge something given (the ontic status of modal aspects) and the creative task assigned to the mathematician to theoretically disclose and deepen this given meaning.

In a general sense this distinction applies to all of mathematics, because mathematics as a special science can be free and creative only when it proceeds from what is ontically given and can be disclosed. Platonism unilaterally reifies what is ontically given, whereas Constructivism unilaterally reifies the human cognitive involvement.

Russell's epigram quoted at the beginning of this article arose from his assessment of an axiomatic approach to mathematics. In such an approach the content of the (terms used in the) axioms is left aside. In addition, the formal relations between the axioms and the deductions made from them in a logically valid way make the issue of truth irrelevant. However, the dream to prove that mathematics is consistent was ruined by Kurt Gödel's famous article (1931) regarding the incompleteness of a consistent system. Later, Hermann Weyl (1970: 269) succinctly summarised Hilbert's disappointment: "It must have been hard on Hilbert, the axiomatist, to acknowledge that the insight of consistency is rather to be attained by intuitive reasoning which is based on evidence and not on axioms".

22. Intuitive insight

The immediate intuitive insight into the uniqueness of number and space in principle exceeds the formalism of any mathematical axiom system and is reflected in the undefined terms of such a system. Through modal abstraction the aspects of number and space are lifted out in order to demarcate the field of investigation of mathematics. Although these modal points of entry fluctuated throughout the history of mathematics between the extremes of an arithmetisation and geometrisation, mathematics as a discipline never escaped from their grip, which is still reflected in the most general discipline of number (algebra) and space (topology). In addition, foundational studies need to scrutinise the implications of a non-reductionist ontology for mathematics, that is, explore the mathematical consequences of assuming the irreducibility and mutual coherence between the aspects of number and space.

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