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World view, philosophy, and the teaching of arithmetic

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Dilthey's emphasis on the relativity of world and life views inspired Spengler to speak of different worlds of number. Yet, within Greek culture, Greek mathematics switched from arithmeticism to a geometrisation of mathematics. Since the Renaissance the ideal of sovereign human reason, which viewed human understanding as the (*a priori* formal) law-giver of nature, gave rise to the notion of construction. Avoiding the stance of both Platonism and constructivism, an acknowledgement of the ontic status of numbers (in their distinctness and succession), accounted for in terms of the distinction between law and subject, illustrates the influence of an underlying world view.

Wêreldbeskouing, filosofie en die onderrig van rekenkunde

Dilthey se klem op die relatiwiteit van wêreldbeskouings het Spengler geïnspireer om van verskillende getalle-wêrelde te praat. Nogtans het daar binne die Griekse kultuur 'n verskuiwing van aritmetisisme na 'n geometrisering by die wiskunde ingetree. Sedert die Renaissance het die ideaal van die soewereine menslike denke, wat die menslike verstand as die (*a priori* formele) wetgewer van die natuur waardeer het, aanleiding tot die idee van konstruksie gegee. Wanneer beide die Platonisme en konstruksionisme vermy word, deur die erkenning van die ontiese status van getalle (in hul onderskeidenheid en suksessie), kan rekenskap gegee word van die onderskeiding tussen wet en subjek, wat die invloed van 'n onderliggende wêreldbeskouing illustreer.

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In this article the author explores for the first time the relationship between world view, constructivism and the teaching of arithmetic. It will be shown that the notion of logical construction, dating back to Hobbes and Kant, influenced the currently dominating trend of constructivism in reflections on the teaching of mathematics. Although some mathematicians may want to underplay the status and influence of intuitionism in mathematics, it cannot be denied that it has a direct link to constructivism. As far as the mathematical status of intuitionism is concerned, Brouwer (1964b: 79) mentioned in respect of the differences between formalism and intuitionism that "mathematical entities recognized by both parties on each side are found satisfying theorems which for the other school are either false, or senseless, or even in a way contradictory".[1] One of the greatest mathematicians of the twentieth century, Hermann Weyl (who left the axiomatic formalism of Hilbert and took sides with the intuitionism of Brouwer), confessed that the foundational problems of mathematics had a profound influence on his mathematical life:

> From this history one thing should be clear: we are less certain than ever about the ultimate foundations of (logic and) mathematics. Like everybody and everything in the world today, we have our 'crisis'. We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively 'safe', and has been a constant drain to the enthusiasm and determination with which I pursued my research work. This experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavors mean in the context of man's whole caring and knowing, suffering and creative existence in the world (Weyl 1946: 13).

The underlying perspective which I employ receives a specific focus in this article, which is different from previous research in related fields, such as defining mathematics (Strauss 2011a), discussing the difference between ordinal and cardinal numbers (Strauss 2006), investigating mathematical notions of continuity (Strauss 2002), analysing the ontic foundations of the logical principle of the excluded middle (Strauss 1991), and exploring the core meaning of number (Strauss 2011).

1 From this quotation it is clear that one cannot simply refer to 'mathematicians' in an undifferentiated way, because those oriented to axiomatic-formalism differ from intuitionists (and logicists such as Russell, Frege, and Gödel).

1. The relativity of world views[2]

At the beginning of the twentieth century some neo-Kantian philosophers viewed philosophy as the theory of world views (*Weltanschauungslehre*).[3] Although the scope of philosophy does include most of what is alleged in a world and life view, the foundational role of philosophy in respect of the various academic disciplines necessarily, in many respects, exceeds what is implicitly or explicitly accounted for in any given world and life view. These limitations of a world and life view, according to Dilthey, follow from its rootedness in life itself (Dilthey 1977: 78). The life of every individual creates its own world because it is the ultimate root of a world view (Dilthey 1977: 79).[4] According to him, each different type of world view, within the limits of our thinking, expresses one side of the universe only. Every world view is true but one-sided. We are not capable of observing all these sides synoptically. We can only observe the Light of Truth in diverse broken beams.[5]

The underlying perspective operative within Dilthey's thought is found in his historicist orientation in terms of which he surrendered to the relativity of all human conditions and ways of believing as the final step towards the liberation of humankind. With this historical consciousness humanity acquires the sovereign power to appropriate the quality of every experience, fully and without any prejudice, giving itself as if no system of philosophy or faith can bind a person. Life becomes free from conceptual knowing; the mind becomes sovereign in respect of all cobwebs of dogmatic thought. Every beauty, every kind of holiness, every sacrifice revived, and explained, opens vistas disclosing a reality.[6]

2 I wish to express my gratitude to anonymous reviewers for critical remarks that prompted alterations and additions to an earlier version of this article.

3 Volume 7 of Wilhelm Dilthey's *Collected Works* (1977) is entitled *Weltanschauungslehre*, *Abhandlugen zur Philosophie der Philosophie*.

4 Die letzte Wurzel der Weltanschauung ist das Leben".

5 "So drückt jede derselben in unseren Denkgrenzen eine Seite des Universums aus. Jede ist hierin wahr. Jede aber ist einseitig. Es ist uns versagt, diese Seitenzusammenzuschauen. Das reine Licht der Wahrheit ist nur in verschieden gebrochenem Strahl für uns zu erblicken" (Dilthey 1977: 224).

6 "Das historische Bewußtsein von der Endlichkeit jeder geschichlichen Erscheinung, jedes menschlichen oder gesellschaftlichen zustandes, von der relativität jeder Art von Glauben ist der letzte Schritt zur Befreiung des Menschen.

2. Are numbers purely cultural products?

The relativism made available by historicism inspired Ostwald Spengler, in his famous work *The decline of the West*, to argue for the relativity of number and number systems. He attempts to claim that number as such does not exist. Spengler believes that there are different worlds of number because there are multiple cultures. According to him, we therefore find Indian, Arabic, Antique, and Western types of number, each with its own distinctive uniqueness and each expressing a different tone of the world and, as an ordering principle, each with a limited symbolical validity. There is therefore more than one instance of mathematics.[7]

With good reason this perspective highlights the fact that different cultures indeed developed different number symbols and different types of number concepts. But at the basis of all these variations one cannot deny a given diversity and plurality. Things are distinct prior to their being identified and distinguished. It is only once the question 'How many?' is asked that a human response is required, and this response results in the use of (culturally determined) numerals (number symbols). Unfortunately, Spengler does not realise that the differences he has in mind are rooted in a shared ontic reality – the quantitative meaning of the one and the many.[8] At this point we will argue that, in response to what is ontically given, human

Mit ihm erreicht der Mensch die Souveränität, jedem Erlebnis seinen Gehalt abzugewinnen, sich ihm ganz hinzugeben, unbefangen, als wäre kein System von Philosophie oder Glauben, das Menschen binden könnte. Das Leben wird frei vom Erkennen durch Begriffe; der Geist wird souverän allen Spinneweben dogmatischen Denkens gegenüber. Jede Schönheit, jeder Heiligkeit, jedes Opfer, nacherlebt und ausgelegt, eröffnet Perspektiven, die eine Realität aufschließen" (Dilthey 1927: 290-1).

7 "Eine Zahl an sich gibt es nicht und kann es nicht geben. Es gibt mehrere Zahlenwelten, weil es mehrere Kulturen gibt. Wir finden einen Indischen, Arabischen, antiken, abendländischen Zahlentypus, jeder von Grund aus etwas Eignes und Einziges, jeder Ausdruck eines anderen Weltgefühls, jeder Symbol von einer auch wissenschaftlich genau begrenzten Gültigkeit, Prinzip einer Ordoung des Gewordnen, in der sich das tiefste Wesen einer einzigen und keiner andern Seele spiegels, derjenige, welche Mittelpunkt gerade dieser und keiner anderen Kultur ist. Es gibt demnach mehr als eine Mathematik" (Spengler 1923-I: 78-9).

8 The Greek word '*on*' designates what *exists*.

mathematical reflection cannot escape from the prevailing cultural conditions of a specific society.

However, the position taken by Spengler cannot account for different orientations within one and the same culture. On the one hand, robust claims regarding historical relativity and historical change never manage to avoid some form of universality. Amidst the threat of relativism introduced by Dilthey's historicism, with its implied denial of universality, it is remarkable that Dilthey nonetheless still had to acknowledge one universally valid truth, namely an ordering according to laws.[9]

3. Constant flux: universality between thought and being

The early Greek philosopher Parmenides already identified thought and being, while Heraclitus asserted the existence of an ontic order designated with the term *logos* (world order – *cf* Diels-Kranz 1959, B-Fr.30 ff). In Greek culture we find one of the first clear examples of the way in which a world and life view influenced scientific endeavours. Greek philosophy wrestled with the awareness of the persistence of entities (the fact that entities endure over time). This awareness was challenged by the notion of change. Heraclitus asserted that whatever there is, is in constant flux (Diels-Kranz B Fr.90). Of course, this assertion does not notice the subtle presence of the term 'constant' in the expression 'constant flux'. With the intention of distinguishing between a law and what is governed by such a law, Avey is therefore justified in questioning relativism: "There is, however, another aspect of Heraclitean philosophy which should not be ignored, and which relativist theory does not always find it convenient to emphasize. The law of change does not itself undergo change in the manner of the changing particulars" (Avey 1929: 521). This shortcoming continued to plague all those thinkers who wanted to emphasise change at the cost of constancy. The ultimate struggle with the changefulness of the universe and the urge towards what is immutable and constant came to expression in the dualistic opposition of matter and form.

9 "die Ordnung nach Gesetzen; diese ist die einzige Wahrheit, die uns allgemeingültig gegeben ist" (Dilthey 1977: 224).

According to Dooyeweerd, this dualism manifested itself in a dialectical movement in which primacy is alternatively given to either of these poles. While the initial phase of Greek philosophy advanced by ascribing primacy to the matter motive, the motive of changefulness, from Anaxagoras to Socrates and Plato, we increasingly witness how the primacy was shifted back to the form motive (Dooyeweerd 2004). Early Greek mathematics appreciated number as the key to an understanding of the universe, but through the discovery of irrational numbers it switched to geometry and space as an alternative mode of explanation. This shift demonstrates how the world view tension between constancy and change (form and matter) directed the entire development of Greek mathematics, from its initial arithmetisation to its subsequent geometrisation.

One of the most powerful elements in this Greek legacy is found in Aritstotle's conviction that infinity and continuity merely have a 'potential' existence and therefore can never be completed. It was practically never questioned by nearly all mathematicians and philosophers (Becker 1964: 41) until Georg Cantor developed his set theory and transfinite arithmetic between 1874 and 1899.

From a historical point of view, Greek mathematics contradicts Spengler's claim, namely that every culture as a whole has its own type of number, for this development within one culture opted for two different positions – arithmetical and spatial. The switch from the perspective of number to that of space reflects the realisation that, although every numerical relationship (fraction) allows for a geometric representation, not every relationship between two line stretches can be represented in a numerical way, that is, by means of rational numbers. As a result, geometry assumed a dominant position in respect of arithmetic, explaining why Euclid treated numbers as a part of geometry (Laugwitz 1986: 9). Inspired by the motive of form, measure and harmony, Plato developed his theory of transcendent, eternal and immutable ontic forms (*eidè*). Aristotle transposed these ideas into the universal substantial forms of concrete entities – the latter being conceived as the union of matter and form. That this union is constitutive of every substance generated a view dominating the ensuing medieval metaphysics of being (also known as the 'chain of being').

4. Modernity: the sovereignty of thought and functionalism

Since Descartes modern mathematics once more started to pursue an *arithmeticistic* path, culminating in the work of Bolzano, Weierstrass, Dedekind and Cantor in the nineteenth century. This development was accompanied by a switch from a substantialistic mode of thinking to a functional approach. Function concepts or concepts of relation are geared towards the how of things and events and are therefore not concerned with their concrete what. The neo-Kantian thinker, Heinrich Rickert, aims at binding the natural sciences to the ideal of transforming all thing concepts into concepts of function (explicitly designated as concepts of relations). This view continues the aim to reduce the entire universe to one or another mode of explanation (function or relation). According to Rickert (1913: 68-70), the (functionalistic) logical ideal of the natural sciences finds its limit in the uniqueness (individuality) of experiential reality itself.[10]

This entire development since the Renaissance breathed the spirit of a new world and life view, namely that of modern Humanism. This world and life view no longer accepted an objective world order, because primacy was given to subjective human thought. The German physicist Von Weizsäcker writes:

> This state of affairs is characteristic of modernity. It is not the world in which I find myself that guarantees my existence. This guarantee is not lost, for when I recover the world then it is as the object of my self-assured thinking, that is to say, as an object which I can manipulate.[11]

From its inception modernity attempted to reduce all of reality to one or another principle of explanation, such as movement (the mechanistic main tendency of classical physics up to Hertz – by the end of the nineteenth century), or thinking, as Descartes asserts with his claim: "at all events it is certain that I seem to see light, hear a noise,

10 Functionalism reduces entities to functions, while substantialism reduces functions to entities.

11 "Dies ist ein charakteristisch neuzeitlicher Sachverhalt, Nicht die Welt, in der ich mich vorfinde, garantiert mein Dasein. Diese Garantie geht nicht verloren, und wenn ich die Welt wiederfinde, dann als Gegenstand meines selbstgewissen Denkens und darum als Objekt, das ich hantieren kann" (Von Weizsäcker 2002: 130-1).

and feel heat; this cannot be false, and this is what in me is properly called perceiving (sentire), which is nothing else than thinking" (*Meditation* II). Alternatively, Hume holds: "To hate, to love, to think, to feel, to see; all this is nothing but to perceive" (*A Treatise of Human Nature*, Book I, Part II, Sec vi).[12]

One of the important consequences of this new humanistic world and life view is that it denied the real existence (ontic nature) of number by claiming that number is merely a mode of thinking.[13] The implication of this world and life view is that the meaning of number is viewed as a pure mental construct. The implicit assumption behind this view is found in the notion that human conceptual thinking assumes the role of lawgiver. In the eighteenth century Immanuel Kant advanced this view, claiming that human understanding creates its laws (a priori) not out of nature, but prescribes them to nature (Kant 1783, II:320; § 36).[14] This view supports the conviction that the world has a rational structure. In fact, it is merely intelligible but not rational. The motive of natural scientific control also informed E T Bell's view in his well-known work *Men of mathematics*: "If 'Number rules the universe' as Pythagoras asserted, then number is merely our delegate to the throne, for we rule Number" (Bell 1965-I: 16).

12 Of course, in addition to the thinking substance (*res cogitans*), Descartes accepted an extended (corporeal) substance (*res extensa*). Ever since the Greek geometrisation of mathematics material things were characterised by extension as their supposed essential feature. Regarding the latter, Descartes (1965a: 200 – Part I, IV) writes: "That the nature of body consists not in weight, hardness, colour, and the like, but in extension alone". Kant's characterisation of material bodies is also oriented toward space. When our understanding leaves aside everything accompanying their representation, such as substance, force, divisibility, and so on, and likewise also separates that which belongs to sensation, such as impenetrability, hardness, colour, and so on, then this empirical intuition leaves something else, namely extension and shape. "So, wenn ich von der Vorstellung eines Körpers das, was der Verstand davon denkt, als Substanz, Kraft, Teilbarkeit usw., imgleichen, was davon zur Empfindung gehört, als Undurchdringlichkeit, Härte, Farbe usw. absondere, so bleibt mir aus dieser empirischen Anschauung noch etwas *übrig*, nämlich Ausdehnung und Gestalt" (Kant 1781/1787-B:35).

13 Descartes believes that "number and all universals are mere modes of thought" (*Principles of Philosophy*, Part I, LVII).

14 A similar view is found in the thought of Piaget. Njisane (1992: 27) explains that to "Piaget knowledge is constructed as the learner strives to organize his experiences in terms of pre-existing mental structures and schemes". *Cf* also von Glasersfeld 1996: Chapter 3.

The mere fact that 'laws' are mentioned reveals that modern Humanism went through Christianity, since the notion of law and subject traditionally forms part of the Christian distinction between Creator and creation. The ultimate world and life view question in this instance concerns the status of law: is it God-given or is it a human construct? Within the philosophy of mathematics this question is related to another issue, namely whether mathematical truths are contained in a (transcendent) world independent of the thinking human subject (Platonism), or whether mathematical entities are constructed through the intellectual endeavours of mathematicians (constructivism)? Are numbers something 'objective' or 'subjective'?

5. The core ontic meaning of number

Only when the term existence is restricted to the reality of concrete entities, such as material things, plants animals, human beings and cultural artefacts, are we misguided into denying the ontic status of the quantitative aspect of reality by transposing it 'into' the human 'mind'. Yet the crucial question is: are there not rather, prior to any human response or construction, a given multiplicity of entities and a given multiplicity of aspects or functions of reality? In his biographical work on Kurt Gödel the mathematician Hao Wang points out that Gödel is very "fond of an observation that he attributes to Bernays": "That the flower has five petals is as much part of objective reality as that its color is red" (Wang 1988: 202).

This approach suggests that the quantitative aspect of things (entities) is not a product of thought – human reflection can at most explore this given (functional) trait of reality by analysing what is entailed in the meaning of multiplicity (the one and the many). But, in doing this, theoretical and non-theoretical thought merely explore the given meaning of this quantitative aspect in various ways, normally first by forming (usually called 'creating') number words such as 'one', 'two', 'three', and so on. The simplest act of counting already had to explore the original meaning of the quantitative aspect of reality. This happened in a twofold manner: every successive number word ('one', 'two', 'three', and so on) or number symbol (numerals, such as '1', '2', '3', and so on) is correlated with whatever is counted. Those involved in teaching counting at school hold that normal children learn to count quite easily, at least if they are frequently given the opportunity to count within normal everyday situations (Murray 1992: 258-9). Children learning to count should not be encouraged to count 'in-the-abstract' before they have mastered the ability to "count out"; in other words, the ability to count a number of items correctly (Murray 1992: 253). Murray also points out that "true counting situations" are intimately connected to similarities and differences. This entails that the "counting aspect" must be the "*main* feature of the situation", that is to say, when a child learns to count, the items to be counted should not be too different from each other. In the case of four similar slices of bread on a plate the similarities will be dominant, but "if two of the slices of bread had jam on and the other two nothing, this *difference* would be of more interest to the child than the total number of slices" (Murray 1992: 252).

Suppose we level the playing ground by 'balancing' the differences and similarities, such as when we look at a white cat and a black cat. In this instance, it may be equally striking for a child to notice the colour difference and the cat-ness similarity. However, as Frege clearly perceived, noticing that both cats share the feature of 'being a cat' requires that all particulars (such as being black or being white) are disregarded. Does this mean that one can arrive at the number 2 by abstracting from the fact that the one cat is black and the other one white? Frege uses this example in order to highlight the shortcomings of abstraction: "The concept 'cat', that has been obtained through abstraction does indeed contain no particulars, but precisely for that reason it is only one concept" (Frege 1884: 45-6, §34, translation by Dummett 1995: 84). According to him, it must be obvious that the properties through which entities distinguish themselves from each other are indifferent with regard to their number (Frege 1884: 40 ff).[15]

The alternative question to be asked concerns the possibility of distinguishing between diverse modal or functional properties of one and the same entity? In terms of Frege's example of the moon, we may

15 He explicitly asks the question from what one should abstract in order to arrive at the number 'one' when one starts with the moon as an entity. By abstraction, he proceeds with his argument, one only arrives at (more general) concepts such as: 'attendant of the earth', 'attendant of a planet', 'celestial body without its own light', 'celestial body', 'body', 'object' (*Gegenstand*) – and nowhere in this series the number '1' will occur (Frege 1884: 57, §44).

be more specific: does the moon have any quantitative properties? Frege did realise that number is an answer to the question 'how many?' and even explicitly discusses this question in connection with the moon (Frege 1884: 57, §44). He asks whether or not the moon is 'one' or 'more than one'? Yet the legacy of denying the ontic status of the quantitative aspect prevented him from relating numerical properties of entities to what may be called the universal ontic nature of the quantitative aspect of reality. As a result, he categorically denies that "number [is the] property of something" (Frege 1884: 63, §51).

In his remark relating to Cantor's definition of a subset (Cantor 1962: 282), Zermelo also refers to the attempt to introduce the notion of 'cardinal number' with the aid of a process of 'abstraction', which would imply that a cardinal number is to be regarded as a 'set composed of pure ones'. The cardinality or power of a set disregards any order-relation between its elements. When such an order relation is borne in mind, ordinal numbers are at stake. Counting the 'first', the 'second' and so on therefore employs ordinal numbers. Cantor holds that the concept of a cardinal number emerges when we abstract from the character of the different elements of a given set M and also disregards the order in which they are given (Cantor 1962: 282). Note that there is a difference between a 'given order' and the impossibility of the absence of any order. Zermelo considers it to be important to realise that from the fact that these 'ones' are still mutually distinct it follows that they simply provide the elements of a newly introduced set equivalent to the first one, which means that the required abstraction did not help us (cf his remark in Cantor 1962: 351).

In the example of the slices of bread two features are present at once: selecting what is similar (the slices with jam) and counting what has been selected (there are two of them). The second element highlights the ontic dimension of concretely existing entities (and processes), while the first one depicts the ontic dimension of the various modes of being (modalities, aspects, functions) of reality. In our everyday experience counting always concerns both of these dimensions, for the first (entitary directed) question is: what is counted? whereas the second pertains to the (modal) quantitative question: how many are there? Every counted (or countable) specimen is similar (*gleich*) to every other one in the numerical sense of just being another 'one' to be counted. An entitary perspective on the previous sentence yields the unified 'what concept', while a modal functional arithmetical perspective on it generates the 'number concept' of how many entities there are. In other words, the conceptual identity of a multiplicity of entities cannot eliminate this (modal, functional) numerical multiplicity. Although he is not acquainted with the distinction between the entitary dimension and the dimension of modal aspects, Tait has a very clear understanding of the above issues. He claims that Frege tends to confuse the following two questions: "What are the things to which number applies?" and "What are numbers?" (Tait 2005: 241). In order to highlight the difference between the what and the how (the difference between entities and aspects), we prefer to phrase the second question in terms of how by asking "How many are here" instead of "what are numbers"? A theoretically articulated understanding of the meaning of number is therefore only possible on the basis of abstracting the quantitative modal aspect (something denied by Frege). Such an understanding is therefore an instance of modal abstraction.

Yet there is an even more fundamental issue at stake, because in our reference to number words and number symbols (numerals) we noted that the question 'how many?' requires a human response. The question is: Are there (universal) ontic features presupposed in our answer to this question, which are quantitative in nature? Or, alternatively: Do we have to revert to the position that number and all universals are creations of the human mind (as Descartes asserted)?

Bearing in mind that, to Cassirer, 'relational concepts' are similar to function concepts, his following statement is significant for the distinction between aspects (functions) and entities (things, so-called 'objects'). Cassirer highlights what we have called aspectual abstraction or rather modal abstraction.

> The function of 'number' is, in its meaning, independent of the factual diversity of the objects which are enumerated. [...] Here abstraction [...] means logical concentration on the relational connection as such (Cassirer 1953: 39).

The theory of modal aspects indeed reveals a new avenue in this respect because it pertains to the 'relational connection' as such. For this reason, the phrase we have used, namely 'the quantitative aspect of reality', implicitly refers to an alternative view of the world, which is foreign to Cassirer's thought and to Frege's understanding of the nature of number, because, according to the latter, "specifying a number contains a statement about a concept" (Frege 1884: §74, page 81).

A multiplicity of so-called ones does not produce a number. Although apples may be counted one-by-one, it is the uniqueness of any number of them that reveals something of the distinct position of each succcessive number en-count-ered in the process of counting. In the context of such a counting order of succession, every number is either equal to, smaller, or larger than every other number within the system of successive numbers.

Whereas any point in space is as good as another one, every number occupies a unique place or position in the number system. This way of addressing the issues reveals an unavoidable complication entailed in the analysis of the meaning of number, namely that such an analysis of the meaning of number cannot be accomplished in a purely numerical way. This explains why, even if unnoticed, the preceding explanation in fact employed spatial terms in our analysis of the quantitative meaning of number, such as place and position. To give another example: it is common use to refer to the infinitely large and the infinitely small. Yet we do not realise that the large-small opposition as such has a spatial meaning and not a numerical one. The numerical equivalent of this opposition is more-less (many-few). Strictly adhering to the meaning of number therefore would have had to result in fairly unusual formulations, such as infinitely many and infinitely few. The habit of mathematical logic to speak of constants and variables does not realise that these terms are derived from the core meaning of the kinematic and physical aspects.

6. Complications for our understanding of the number system

In connection with a child counting four slices of bread, Murray employed the phrase "the total number of slices". A closer appraisal of the expression "total number" immediately reveals that the primitive awareness of multiplicity and discreteness does not entail the notion of a totality. The term 'totality' is merely a synonym of wholeness and the intuitionistic trend in modern mathematics holds that the whole-parts relation is charactertistic of continuity (or the continuum) (cf Weyl 1966: 84).[16] The notion of a totality flows from thinking of a multiplicity as a 'one'. Obojska explains the difference between Cantor, who viewed a set as "a many considered as a one" and the mereological approach of Lesniewski, who defines the 'part' in terms of the 'whole' (Obojska 2007: 644).

The indebtedness to the use of terms derived from the aspect of space is equally clear in a phrase contained in the second subheading of the above chapter written by Murray. The heading reads: 'The Positional Number System we use'. The term 'system' incorporates both numerical and spatial elements, for it embodies a multiplicity of elements and parts and it designates a structured whole. Whenever a multiplicity is thought together or is united into a whole, we may also speak of a system. Referring to the Positional Number System therefore includes the term number and the other two terms (positional and system) are derived from the meaning of space.[17] Therefore, the meaning of number can only be analysed in a complex way, by employing terms derived from aspects that differ from the numerical aspect.[18]Once a child has mastered what Murray designated as "counting out", it is possible to proceed to a better understanding of what Murray calls the "repeating structure" of number words in counting (Murray 1992: 258). From a historical perspective this "repeating structure" expresses the normal succession of (the natural) numbers, 1, 2, 3, 4, ... which was constantly acknowledged as such. Apart from the succession of natural numbers, no account of their uniqueness (unique position or place) can be given. Of course, on the level of a world and life view and of a philosophical orientation, the crucial question is whether it is the "repeating structure" of number

16 Strauss (2009: 60-1, 301-3, 306-7, 353-5, 391-2, 406-8, 514-5) argues that the whole-parts relation, within different contexts, is equivalent to the original meaning of continuous extension.

17 The meaning of the term 'system' always points at a whole with its (interacting or inter-dependent) parts.

18 The terms 'constants' and 'variables', which play a crucial role in mathematical logic, illustrate the use of key kinematical and physical terms, since they analogically echo the phoronomic meaning of uniform motion and the physical meaning of dynamic change. Where Paul Lorenzen (1976: 1 ff) distinguishes four units of measurement, it reflects these first four aspects of reality (number, space, movement, and the physical): mass, length, duration and charge.

words that creates their succession, or whether the latter is made possible by an ontic order of succession. The fundamental importance of what may be called the numerical order of succession expresses the core meaning of number (which is indefineable) intuitively. This core meaning is particularly acknowledged by the intuitionist trend in modern mathematics. It serves as the foundation for the principle of (mathematical) induction. This principle escapes every attempt to formalise it. Hermann Weyl, a student of David Hilbert, who left the latter's school of axiomatic formalism in support of the intuitionistic mathematics of L E J Brouwer, correctly points out that induction safeguards mathematics from becoming an immense tautology and at once gives a synthetic (non-analytic) character to the assertions of mathematics (Weyl 1966: 86). Skolem (1979: 70) maintains that this also applies to set theory:

> Those engaged in doing set theory are normally convinced that the concept of an integer ought to be defined and that complete induction must be proved. Yet it is clear that one cannot define or provide an endless foundation; sooner or later one encounters what is indefinable or unprovable. Then the only option is to ensure that the first starting points are immediately clear, natural and beyond doubt. The concept of an integer and the inferences by induction meet this condition, but it is definitely not met by the set theoretic axioms such as those of Zermelo or similer ones. If one wishes to derive the former concepts from the latter, then the set theoretic concepts ought to be simpler and employing them then ought to be more certain than working with complete induction – but this contradict the real state of affairs totally.[19]

One of the leading mathematicians of the late nineteenth century, Leopold Kronecker, is notable for his remark that God created the integers (whole numbers) and that everything else is the result of

19 "Die Mengentheoretiker sind gewöhnlich der Ansicht, dass der Begriff der ganzen Zahl definiert werden soll, und die vollständige Induktion bewiesen werden soll. Es ist aber klar, dass man nicht ins Unendliche definieren oder begründen kann; früher oder später kommt man zu dem nicht weiter Definierbaren bzw. Beweisbaren. Es ist dann nur darum zu tun, dass die ersten Anfangsgründe etwas unmittelbar Klares, Natürliches und Unzweifelhaftes sind. Diese Bedingung ist für den Begriff der ganzen Zahl und die Induktionsschlüsse erfüllt, aber entschieden nicht erfüllt für mengentheoretische Axiome der Zermelo'schen Art oder ähnliches; sollte man die Zurückführung der ersteren Begriffe auf die letzteren anerkennen, so müssten die mengentheoretischen Begriffe einfacher sein und das Denken mit ihnen unzweifelhafter als die vollständige Induktion, aber das läuft dem wirklichen Sachverhalt gänzlich zuwider."

human endeavours ("Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk").[20] Today mathematicians would rather speak about natural numbers, namely (0), 1, 2, 3, ... [21] The emphasis on the key role of natural numbers and integers inspired Kronecker to explicitly introduce the ideal of arithmetisation:

> I also believe that eventually we will succeed in 'arithmetizing' the entire content of mathematical disciplines, that is to say, to provide a foundation for it purely and solely on the number concept, taken in the strictest sense (Cabillon 2011).[22]

Kronecker wrote something similar to Cantor from 'Kammer am Attersee' (21 August 1884), when he stated that his starting point is that everything in pure mathematics could be reduced to the theory of the integers and that he believes that it will be possible in all respects.[23]

7. At the crossroad: between discovery and construction

Kronecker is indeed a mathematician who contributed to what ultimately became known as constructivism in mathematics. Poincaré continued the Kantian legacy by distinguishing between logic and intuition. In his famous presentation on the infinite, David Hilbert reminds us that Kant already taught that mathematics disposes over a content which is independent of all logic and therefore can never be based solely on logic (Hilbert 1925: 171). He also points out:

20 In 2005 Stephen Hawking acted as editor of a work which partially used these words of Kronecker: *God created the integers, the mathematical breakthroughs that changed history* (Hawking 2005). A photocopy of the original 1887-article is in my possession.

21 In mathematics the number 0 plays an important role, *inter alia* as additive identity of the system of integers. If a set has the operation of addition, then 0 could be added to any element *x* of it without changing it.

22 "... ich glaube auch, dass es dereinst gelingen wird, den gesammten Inhalt aller dieser mathematischen Disciplinen zu 'arithmetisiren', d.h. einzig und allein auf den im engsten Sinne genommenen Zahlbegriff zu gründen, also die Modificationen und Erweiterungen dieses Begriffs wieder abzustreifen" (Kronecker 1887: 265). Cabillon J G 2011. http://mathforum.org/kb/message.jspa?messageID=1178416&tstart=0, quoting Kronecker on arithmetisation.

23 "Ich bin deshalb darauf ausgegangen, Alles in der reinen Mathematik auf die Lehre von den ganzen Zahlen zurükzuführen, und ich glaube, dass dies durchweg gelingen wird" (Meschkowski 1967: 238).

> Only when we analyze attentively do we realize that in presenting the laws of logic we already have had to employ certain arithmetical basic concepts, for example the concept of a set and partially also the concept of number, particularly as cardinal number [*Anzahl*]. Here we end up in a vicious circle and in order to avoid paradoxes it is necessary to come to a partially simultaneous development of the laws of logic and arithmetic (Hilbert 1970: 199, cf also Quine 1970: 88).

Speaking of 'the laws of logic and arithmetic' underscores the underlying problem running through our fore-going reflections: are these laws creations of human thought or are they rather discovered? In addition, are there perhaps other options?

The intuitive understanding of a law is that it delimits and determines whatever is subjected to it. To the constructivist it appears that the mathematician posits the stipulations or conditions of specific mathematical structures. These stipulations or conditions serve as the laws holding for what is correlated with them as subjects. Surely the conditions holding for something can never coincide with it. The conditions for being green are not themselves green, just as little as the physical laws for matter are themselves material. What is normally viewed as a mere fact – such as stating that 3+4=7 – actually relates certain numbers in a lawful way, conforming to the arithmetical law (operation) of addition. Although self-evident, it must be noted that the statement that 3+4=7 is a numerical (arithmetical) fact. However, in order to appreciate why this is not totally self-evident, we merely have to mention a similar sum – one obtained by first walking 3 miles north and then 4 miles east – in which case one would be 5 miles away from one's starting point. A vector is known to have both direction and distance. This explains at once that it is a spatial subject (a specific line-stretch) and not merely an arithmetical subject (like numbers).

Clearly, numerical subjects ought to be distinguished from spatial subjects, and this observation entails that we now have two different kinds of facts at hand: a numerical fact (designated as 3+4=7) and a geometrical fact (designated as 3 +4 = 5) (Figure 1).

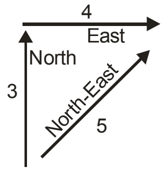

Figure 1

These two facts can only be distinguished on the basis of a prior distinction, namely that between the quantitative aspect and the spatial aspect. The identification and distinguishing of any aspect represents an act of lifting out while disregarding; in other words, an act of abstraction. Since analysis rests on the legs of identification and distinction, which are equivalent to the two legs of abstraction, namely lifting out and disregarding, it follows that abstraction and analysis are synonyms.[24] The human subject always acts while observing, *inter alia*, the normative meaning of logical analysis and the force of the underlying cosmological principle of the excluded antinomy.[25] Nonetheless, the classical (rationalistic) science ideal of modernity is still very much alive when Fern differentiates between morality (dependent upon 'subjective faith') and 'mathematics' and 'science' "that allows us to prove – establish beyond a reasonable doubt in universally accessible, rationally compelling terms":

> The frightening thought, so far as moral convictions go, is that we cannot get around the element of subjectivity, that in the end it all comes down to what we see (or fail to see) in the wolf's eye. In part, I want to allow that this is so; there is no method, scientific or otherwise, that allows us to prove – establish beyond a reasonable doubt in universally accessible, rationally compelling terms – that a particular moral outlook is correct. In this sense, morality rests on faith in a way not true of mathematics, logic or, even, at its core,

24 An extensive argument why modal abstraction is to be regarded as the distinctive feature of scholarly (scientific) thinking is found in Strauss 2009: 45-60.

25 From the fact that our experiential world displays a multiplicity of irreducible modal aspects, it follows that every attempt to reduce an aspect to a different one will result in a clash of mutually irreducible laws, in other words, antinomies. The general assumption of this approach is given in the aim to avoid every reductionistic ism (such as arithmeticism, holism, physicalism, vitalism, psychologism, logicism, historicism, legalism, moralism, and so on), also characterised as a non-reductionist ontology.

> modern science – the affirmation of which serves as a measure of one's basic reasonableness (Fern 2002: 95).

Nonetheless the reality of twentieth-century mathematics tells a different story. It portrays a proliferation of viewpoints which do not merely occur in the philosophy of mathematics, but within the confines of this discipline itself – as Beth and Brouwer confirm. Beth states:

> It is clear that intuitionistic mathematics is not merely that part of classical mathematics which would remain if one removed certain methods not acceptable to the intuitionists. On the contrary, intuitionistic mathematics replaces those methods by other ones that lead to results which find no counterpart in classical mathematics (Beth 1965: 89).

Brouwer (1964b: 78) holds "that classical analysis [...] has less mathematical truth than intuitionistic analysis". He then proceeds with a characterisation of formalism and intuitionism:

> As a matter of course also the languages of the two mathematical schools diverge. And even in those mathematical theories which are covered by a neutral language, i.e. by a language understandable on both sides, either school operates with mathematical entities not recognized by the other one: there are intuitionist structures which cannot be fitted into any classical logical frame, and there are classical arguments not applying to any introspective image. Likewise, in the theories mentioned, mathematical entities recognized by both parties on each side are found satisfying theorems which for the other school are either false, or senseless, or even in a way contradictory. In particular, theorems holding in intuitionism, but not in classical mathematics, often originate from the circumstance that for mathematical entities belonging to a certain species, the possession of a certain property imposes a special character on their way of development from the basic intuition, and that from this special character of their way of development from the basic intuition, properties ensue which for classical mathematics are false. A striking example is the intuitionist theorem that a full function of the unity continuum, i.e. a function assigning a real number to every non-negative real number not exceeding unity, is necessarily uniformly continuous (Brouwer 1964b: 79).

Of course, the reference to addition in distinguishing the above-mentioned facts could be embedded within modern mathematical set theory which normally approaches this domain in terms of the algebraic structure of fields – where the (binary) operations called addition (+) and multiplication (.) conform to the field axioms which are explicitly specified as laws.[26] The fact that addition and multiplication within a system of numbers yield numbers belonging to the initial set is also mathematically articulated by saying that the system of numbers under consideration is closed under the operations (laws) of addition and multiplication. Viewed from the perspective of the strict correlation of law and subject, we may explore the meaning of numerical subjects in the following way. A distinction may be drawn between a set and a system with the aid of two subscripts: s = system; t = set. While a set (in Zermelo-Fraenkel set theory) contains members, that is, numerical subjects such as the natural numbers,[27] a system of number embraces both laws and subjects. The system of natural numbers N_s finds its determination and delimitation in the operations of addition and multiplication. These operations or laws are correlated with the set of natural numbers. This means that adding or multiplying any two natural numbers will always yield another natural number.[28] In this instance, we have, on the law side of the numerical aspect, the arithmetical laws of addition and multiplication, symbolised as (+, ×). Correlated with these laws, in the sense of being determined and delimited by them, we find the numerical subjects, namely the set of natural numbers $_t = (1, 2, 3, \ldots)$ – and united in an encompassing perspective we have the system of natural numbers: $_s\,[(+,\times)$ and $(1, 2, 3, \ldots)]$.

Introducing further arithmetical laws or operations will invariably call for additional (correlated) numbers that are factually subjected to

26 A field is defined as a set *F* such that for every pair of elements *a*, *b* the sum *a*+*b* and the product *ab* are still elements of *F* subject to the associative and commutative laws for addition and multiplication, and combined to the presence of a zero element and a unit (or identity) element (*cf* Bartle 1964: 28, Berberian 1994: 1 ff). This definition of a field is then expanded to that of an ordered field and it is finally combined with the idea of completeness.

27 The 'empty set' is still (negatively) defined in terms of members – it has no members.

28 As noted earlier, the ultimate presupposition of these operations is found in the numerical order of succession. The latter is primitive and is expressed in the principle of induction. The Peano axioms (for the positive integers) yield a mathematical articulation of this primitive arithmetical order of succession. The correlation of the operations of addition and multiplication and their delimiting and determining role in respect of numerical subjects are consistent with Peano's axioms because they are entailed in the complete ordered field of real numbers (*cf* Berberian 1994: 230).

these new determining and delimiting arithmetical laws. For example, if the operation of subtraction is added to those of addition and multiplication, the correlating set of integers ($_t$) is constituted – and considered in their correlation this yields the system of integers $_s$ [(+,×,–) and (... -3, -2, -1, 0, 1, 2, 3, ...)]. Extending the scope of laws therefore continues to expand the scope of numerical subjects. The next step would be to consider the inverse of multiplication, namely division, which expanded the numerical subjects to include the set $_t$ of rational numbers, yielding the system of rational numbers $_s$ [(+,×,–,÷) and $_t$ = (a/b; a,b $\in$ $_t$ / b≠0)]. The preceding explanation is formally similar to the introduction of negative numbers and fractions by Klein.[29]

Once the unbreakable correlation between law side and factual side is acknowledged, the systematic arithmetical statement 3+4=7 can no longer be designated as a 'pure fact' because the factual relation between the numbers 3, 4, and 7 displays the measure of the arithmetical law of addition. As a result, we rather have to refer to a law-conformative (arithmetical) fact displaying an orderliness or lawfulness while meeting the conditions of arithmetical laws.

When we contemplate all the laws pertaining to the sphere of number, we may designate this aspect, according to its law side, as a law sphere. The idea of a modal aspect comprises the acknowledgment of its core meaning (also designated as its meaning-nucleus), its law side and factual side, subject-subject relations at the factual side, and analogical structural features reflection the coherence of a specific aspect with other aspects.[30]

29 In passing, it may be noted that it is not a matter of mathematical maturation enabling mathematical students to discover that there "are numbers between the whole numbers", as Leake asserts (*cf* Leake 1995: 46). Moving from the system of integers to the system of rational numbers (and eventually the system of real numbers) is obtained through a disclosure or deepening of our understanding of the core meaning of the numerical aspect and its inter-modal (anticipatory) coherence with the spatial aspect. Klein also does this by reversing the operations of addition and multiplication (Klein 1932: 23 ff & 29 ff)

30 Within all the post-arithmetical aspects there are also, in addition to subject-subject relations, subject-object relations. Points, for example, are spatial objects, dependent upon a line as spatial subject. Hilbert's axiomatisation of geometry is accomplished on the basis of the spatial subject-object relation. The term 'line' represents the basic existence of a (one-dimensional) spatial subject; the

The core meaning of number, discrete quantity, pertains to all kinds of numbers, including the natural numbers, integers, fractions, real numbers, and imaginary numbers – amply confirmed by Cantor's circumscription of a set as being constituted by clearly distinct (*wohlunterschiedenen*) objects bound together into a whole (*Ganzheit*) (Cantor 1962: 282).[31] The distinctness of the natural numbers reveals the numerical order of succession. However, at this point we must add another specification which is related to the idea of time. Time is not merely or purely physical in nature because we are acquainted with different modes of time. Kant mentioned the first three, namely succession, simultaneity and persistence (Kant 1787-B:219). Whereas the time order within these aspects (the numerical, spatial and kinematic) is reversible, it is irreversible in the fourth mode, the physical. When this perspective of time is incorporated in our understanding of the quantitative aspect it is clear that on the law side of this aspect we must refer to the arithmetical time order of succession.[32] Von Glasersfeld (1996: 163) indirectly alludes to the first two modes of time where he holds "that to count and to consider several things contemporaneously are different activities".[33]

8. The numerical time order of succession: law and subject

The terms 'order' and 'law' are synonymous, although order is often used to designate the inner coherence of a multiplicity of laws. A social order or legal order is constituted by a multiplicity of social or

term 'point' accounts for the primary existence of a (one-dimensional) spatial object, and the phrase 'lies on' highlights the relation between a spatial subject and spatial object – which means that these three terms reflect the spatial subject-object relation.

31 Note that this circumscription of a set makes an appeal to the core meaning of number (a clearly distinct multiplicity) and the core meaning of space (*Ganzheit* – a whole).

32 Within the spatial aspect we meet a time order of simultaneity, within the kinematic a time order of uniformity and within the physical aspect the irreversible time order of causality – the cause always precedes the effect. The difference between succession and causality is evident from the fact that, although the day succeeds the night and the night the day, neither is the cause of the other (already realised by Kant).

33 Von Glasersfeld derives this insight from Caramuel, *cf* page 170.

jural norms. If these norms were in conflict, the term order could not be applied to them. On the law side of the quantitative aspect, one may discern a multiplicity of arithmatical laws belonging to this law sphere. What makes every multiplicity and every succession possible is the numerical time order of succession. The human (mathematical) response to what is ontically given is found in the articulation and formulation of arithmetical laws and in discerning the quantitative subjects correlated with these laws. Although constructivism in mathematics may credit the thinking mathematical subject with the ability to construct both arithmetical laws and arithmetical subjects, the important point is that it at least acknowledges this strict correlation between law side and factual side.

Myhill, for example, who appreciates Brouwer as the originator of "constructive mathematics",[34] introduces the notion of a 'rule' (the equivalent of what we have designated as 'law side') as "a primitive one in constructive mathematics"; "We therefore take the notion of a rule as an undefined one" (Myhill 1970: 748).[35] In his encompassing introduction to set theory (the third impression), Adolf Fraenkel refers to the peculiar constructive definition of a set which accepts, as a foundation, the concept of law and the concept of natural number as intuitively given.[36] Implicit in this foundation is the acknowledgment of the strict correlation between the law side and the factual side of the numerical aspect. There is something similar in the thought of Cassirer when he discusses the views of Helmholtz and Kronecker. He points out that, fundamental as it is, the concept of order "does not exhaust the whole content of the concept of number" for something new appears when number is "understood and applied as a plurality" (Cassirer 1953: 41). This explanation tacitly assumes the strict correlation of law and subject, for on the law side of the quantitative

34 Troelstra and Van Dalen (1988: 16-33) included a brief history of constructivism in Volume I of their work, *Constructivism in mathematics*.

35 Myhill received his Harvard PhD under W V Quine.

36 "Ohne die Stellung wieterer intuitionistischer Gruppen und anderer Richtungen [...] zum Mengenbegriff zu schildern, sei hier noch auf die wesentlich abweichende Auffassung Brouwers hingewiesen. Dieser stellt eine eigenartige rein *konstruktive* Mengendefinition an der Spitze, bei der der Begriff der *natürliche Zahl* und der des *Gesetzes* als intuitiv gegeben zugrunde gelegt warden" (Faenkel 1928: 237).

mode it discerns order and on the factual side there are multiple numerical subjects ('plurality').

Yet the formulation of arithmetical laws, which often takes the form of stipulating, ultimately depends on the ontic meaning of the quantitative aspect. For this reason, arithmetical laws should also be appreciated as natural laws, on an equal footing, for example, with physical laws (the so-called 'laws of nature', such as the law of energy conservation, of non-decreasing entropy and the law of gravity).[37] They co-condition the existence of physical entities which invariably also function within the numerical aspect. Physicists still appreciate mathematical formulations, with their universal validity, as an essential ingredient of physics. Von Weizsäcker (1993: 113) states: "Laws capable of mathematical formulation finally form the hard core of natural science: not the important detail, but the form of universal validity".[38] From a different discipline we may appreciate a similar statement by the German paleontologist Otto Schindewolf (1993: 5) who remarks:

> Recently, in fact, geologists have in all seriousness raised the question of whether natural laws are indeed constant or whether they are not perhaps subject to change over geologic time. ... If one defines the laws of nature as rules according to which processes always takes place in the same way everywhere, there can naturally be no question of mutability and development over time. It is only our *formulation* of the laws of nature that is mutable. As soon as we learn from experience that the concept of a law is not universally applicable

37 Skolem (1970: 545) implicitly distinguishes between universal modal laws and type laws when he argues that the truth of a mathematical theorem does not need experimentation, adding that "arithmetical proofs by complete induction are quite certain". The mentioned physical laws display modal universality because they hold for all classes of physical entities (this is what Kant had in mind when seeking synthetic concepts *a priori*). Physical type laws, by contrast, hold for a limited class of entities only (such as the law for being-an-atom – cf Strauss 2009: 25-6, 420-1). The amazing rotary motors that drive the bacterial flagellum (up to 100 000 rpm – with a size of 1/100,000th of an inch) are parts of biotic entities determined by type laws (Peterson 2010: 277).

38 In another context, he writes that the quantitative results of astronomy are based on physical laws, and that we postulate, as a working hypothesis, a universal validity for these laws (Von Weizsäcker 1993: 25).

> because it is somehow contingent upon time, then the law should be excluded from the formulation.

The moment we acknowledge the numerical and spatial as aspects of (ontic) reality, displaying both a law side and a factual side, the extremes of Platonism and constructivism can be avoided. Platonism elevates the outcome of mathematical endeavours, manifest in the erection of mathematical structures, mostly designated by metaphors, such as those found in algebra where mention is made of groups, fields, rings, radicals, ideals and so on, to the level of eternal, supra-temporal mathematical truths in themselves. In doing so, it identifies the outcome of mathematical activities with the ontic structure of these aspects. In his well-known presentation on Platonism in mathematics, Paul Bernays (1976: 65) mentions that the application of Platonism in analysis and set theory is so widespread that it is not an exaggeration to state that Platonism reigns in mathematics.

Constructivism, by contrast, discards what is ontically given, namely the aspects of number and space, and accredits subjective human thought with the power of constructing whatever occurs within mathematics. Brouwer accentuates "introspective construction" as the basis of intuitionistic mathematics, which means that mathematics is entirely a construction of the mathematical subject. His starting point is the intuition of "two-oneness" – without contemplating the ontic givenness of multiplicity and succession. This introspective point of departure encompasses both the discrete and the continuous as well as the infinite divisibility of the linear continuum:

> Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the 'between' which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units (Brouwer 1964a: 69).

Whenever modal functions (often designated as 'abstract entities' or 'properties') are contemplated, they are either transposed to a suprasensory 'intelligible realm' (as Platonism did in its various forms, traditionally also known as realism), or they are embedded in the creative powers of the individual (and often collective) human mind (intuitionism in mathematics and other variants of nominalism in philosophy and the various scholarly disciplines). With reference to Gauss, Kronecker continues the Cartesian conviction, namely that the field of investigation of arithmetic is merely a product of human understanding. Kronecker believes that since space and time have a reality outside the human mind, their laws are not completely prescribed in an *a priori* manner.[39]

In passing we may note that the primitive meaning of numerical succession (the foundation of our most basic understanding of infinity in the literal sense of endlessness) cannot be reduced to logic. Logicism had to concede that it failed in providing a successful reduction to logic of the notion of infinity. Myhill (1952: 182) remarks:

> ... the axioms of Principia [Mathematica] do not determine how many individuals there are; the axiom of infinity, which is needed as a hypothesis for the development of mathematics in that system, is neither provable nor refutable therein, i.e., is undecidable.

We add the words of Kline, stating that Hilbert "did agree with Russell and Whitehead that infinite sets should be included. But this required the axiom of infinity and Hilbert like others argued that this is not an axiom of logic" (Kline 1980: 246).

The complications discussed in connection with our understanding of the number system entailed that an analysis of the meaning of number had to employ terms derived from non-numerical aspects, *inter alia*, the aspect of space. We referred to terms such as 'large' and 'small', a 'whole' or 'totality' (the whole-parts relation), 'position' (place), 'domain' and so on. Whereas this complication concerns our description of numbers, a different feature emerges when we consider the way in which different systems of number (analogically) imitate specific spatial traits.[40] We have noted that wholeness is equivalent to continuous extension, implying that speaking of whole numbers in a numerical way imitates the totality character of spatial continuity.

39 "Der principielle Unterschied zwischen der Geometrie und Mechanik einerseits und zwischen den übrigen hier unter der Bezeichnung 'Arithmetik' zusammengefasste mathematische Disciplinen andererseits besteht nach Gauss darin, dass der Gegenstand der letzteren, die Zahl, *bloss* unseres Geistesproduct ist, während der Raum ebenso wie der Zeit auch ausser unserem Geiste eine Realität hat, der wir a priori ihre Gesetze nicht vollständig vorschreiben können" (Kronecker 1887: 265).

40 An analogy points at differences and similarities. To be more precise, an analogy is present when two entities or aspects are similar in that respect in which they differ.

The reverse side of this coin is found in rational numbers (fractions), which imitate the spatial whole-parts relation.[41] The real numbers imitate spatial continuity in the full sense of the word.[42] Whereas this account pertains to inherent numerical properties analogically reflecting spatial features, advancing spatial representations of different types of numbers does the opposite (Leake 1996: 45-53).

9. Conclusion

World view commitments and philosophical orientations continue to exert a direction-giving and systematically articulated influence upon intellectual endeavours, including the developmental history of mathematics and the teaching of mathematics at school. We have investigated this claim from various angles and opted for an alternative which avoids the extremes of mathematical Platonism and constructivism, briefly illustrated as far as the teaching of arithmetic at school is concerned. Since the aspects of reality are intuitively known to us from our early childhood, it is most natural to explore this implicit knowledge in the teaching of mathematics. In fact, the use of an abacus demonstrates the implicit awareness of different aspects of reality, because children immediately notice the colour, motion, shape and multiplicity of the various blocks. They quickly learn how to make simple arithmetical calculations such as adding and subtracting by moving the blocks in the same or opposite directions. The next step is to disregard the non-numerical properties of the blocks such as their colour, shape and the possibility to move them to and fro in order to focus solely on the arithmetical question: how many? Of course, the many-sided experience of children does not stop here because they also experience the abacus in the other aspects of reality – as a cultural artifact (its formative aspect), that it has a name (its function within the sign mode), that it belongs to someone (the jural aspect evinced in the accompanying property right),

41 At the same time, through the imitation of the whole-parts relation, the literal meaning of an infinite succession of numbers (such as the row of natural numbers) is turned ‘inwards’, highlighting one of the features of continuity, namely that it is infinitely divisible.

42 *Cf* Strauss 2011 for a more extensive account of these inter-aspectual connections.

and so on. From their earliest years children are therefore exposed to the identification and distinguishing of different aspects, in other words, to what we have designated as modal abstraction. Of course they are not acquainted with a theoretical account of what modal abstraction entails, just as little as language users are acquainted with a theoretical explanation of what happens when they are speaking. This article is written for scholars, not children, although the distinctions drawn may impact upon the teaching of arithmetic. Exercising the values of intellectual honesty and academic freedom may therefore enhance a practice in which world and life view as well as philosophical presuppositions are articulated and explained more explicitly.

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