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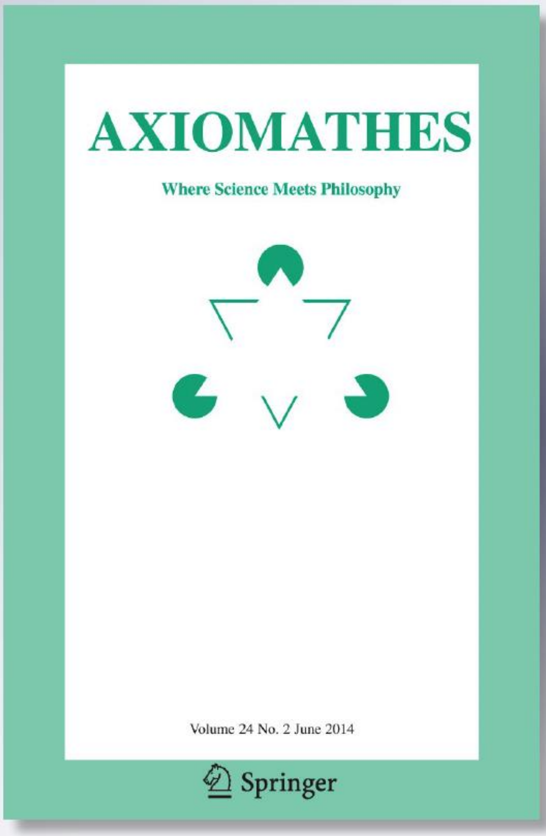

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Abstract

During the history of mathematics, since the discovery of incommensurability (irrational numbers), arithmeticism and geometricism constantly switched roles. Even after the reappearance of arithmeticism during the 19th century we find that Frege, by the end of his life, returned to the view that mathematics is really entirely geometry. More recently 'continuum theorists' in France (Longo, Thom and others) in a similar way opted for the view that the continuum precedes the discrete. After the irreducibility of the spatial whole-parts relation and the spatial order of simultaneity are briefly highlighted, the limitations of human eye-sight in respect of lines and points are explained, focused upon what we can imagine or envisage. Plato's account of knowledge (his theory of transcendent ideal forms) reflects a view on thinking and sensing which is also related to Cantor's view that there are an actually infinite number of entities in the world. The capacities of the human imagination are related to idealized structures, to Euclid's characterization of points and lines, and to the views of Aristotle on (the irreducible continuity of) a line and its points which remained in force until the 19th century. Although Bolzano once more launched an attempt to arithmetize continuity, it appears as if Weierstrass, Cantor and Dedekind finally succeeded to bring this ideal to completion. Their ideas are linked to the number line and critically assessed by analyzing the contradiction present in Grünbaum's attempt to explain the continuum as an aggregate of unextended elements (degenerate intervals). In support of this criticism an appeal is made to the spatial whole-parts relation (Bernays, Brouwer and Weyl) as well as the way in which Bernays acknowledges the uniqueness and mutual coherence between number and space. Finally, a line-stretch is characterized as a one-dimensional spatial subject, given at once in its totality and delimited by two points – but it is neither a *breadthless length* nor the (*shortest*) *distance between two points*.

1. **Background distinctions and perspectives**

Implicit in our everyday experience of the world there are an awareness of distinctions relevant to a proper ‘positioning’ of the question: what is a line? These distinctions will be used to contextualize the question what a line is. Let us start with our experience of reality which is embedded in an awareness of multiple (natural and social) entities as well as numerous aspects or modes of being within which such entities function. For example, identifying couches gives an answer to the question about their ‘what-ness.’ Once we have identified particular things, ‘how-questions’ regarding their various modes or functions or aspects could be asked, such as: how expensive is this couch?; how comfortable is it?; how strong is it?; how large is it?; and how many are there? These questions capture the meaning of different modes of being (modal aspects or ways of existence), *in casu* the economic, sensitive, physical, spatial and numerical. Naturally our focus in this article will be on the last two aspects, *number* and *space*.

Entities which are distinct can always be numbered. By lifting out their quantitative meaning they appear as distinct units which are countable. Discerning their numerical properties therefore has to disregard non-numerical features – but note that they are just disregarded, not *eliminated*. As Tait explains: ‘Distinct units are indeed distinguished by their properties, but when from a set of two cats, one white and one black, we ‘abstract’ the number two as a set of pure units, the units are not white and black, respectively, and they are not cats‘ (Tait, 2005:241). One may relate this meaning of the word ‘abstract’ to the meaning of analysis, which comes to expression in logical acts of identification and distinguishing. The latter, identification and distinguishing, is equivalent to the meaning of abstraction, for abstraction concerns lifting out (identifying) certain properties by *disregarding* others (distinguishing) – on the basis of similarities and differences.

The nature and role of an abacus neatly illustrates what is here at stake. As a calculating device the abacus can play a constructive role in the teaching of arithmetical skills. Consider an elementary abacus where the grouping of beads may represent addition on the levels of ones, tens or hundreds. Initially, the typical many-sidedness of the beads are left intact simply because they form part of the visible image of the abacus and its beads. This means that the beads are observed in their multi-aspectual nature, including their specific color, their typical shapes and sizes, as well as their flexibility (moveable in opposite directions). Because the use of the abacus is directed at enhancing arithmetical skills, these non-arithmetical properties of the abacus are left intact, which makes it easy eventually to abstract from them by lifting out merely and solely the quantitative aspect of the beads. Phrased in terms of what Tait said we can point out that the numerical units obtained through the operations performed with the aid of the abacus do not yield numbers which are spherical, moveable or colored.

This example at once highlights the fact that aspectual (modal) abstraction represents the distinctive feature of scholarly (scientific) thinking (see Strauss, 2009:45-60). Yet, it is insufficient merely to refer to *abstraction*, because our everyday (pre-scientific) concepts of different kinds of entities (such as material things, plants, animals, and cultural objects) already represent a kind of abstraction, one that is directed at different kinds or types of entities. When the modes, aspects or functions of reality are lifted out while disregarding other aspects, *modal abstraction* ensues. Sometimes other designations are found, for example when modal abstraction is seen as idealizing abstraction (Bernays, 1976:37, 196; Diemer, 1970:213).

Of course one of the modes of reality distinguished from the numerical is the spatial aspect. Whereas distinctness, referred to in connection with the meaning of number, suggests the idea of discrete quantity, our awareness of the meaning of space traditionally was related to continuity or to continuous extension. The prime example of something continuous within mathematics, to be more specific, within geometry, is a line-stretch.

Although we shall probe some of the historically significant reflections on continuity below, it should be noted in advance that since Weierstrass, Cantor and Dedekind, the three prominent German mathematicians who produced their most important work during the last part of the 19th century, modern mathematics believes to have bridged the gap between discreteness and continuity. Posy alludes to these developments as a 'period of growing mathematical abstraction' which, through

> analysis and its generalizations in algebra and topology ... took mathematics irrevocably away from any dependence on perceptual intuition. ... mathematicians finally closed the ancient gap between the notion of number and that of a continuous magnitude like a surface or a smooth motion (Posy, 2005:321).

Of course the question is if there is just *one kind* of intuition, namely *perceptual intuition*? Is it not the case that we also have other intuitions, such as a *numerical intuition* and a *spatial intuition*? A prominent mathematician, Bernays (the co-worker of Hilbert), does not hesitate to distinguish between our 'arithmetic' and 'geometric' intuitions (*Anschauungen*). He rejects the widespread view that this distinction (between a numerical intuition and a spatial intuition) concerns time and space, for according to him the proper distinction needed is that between the *discrete* and the *continuous* (Bernays, 1976:81).

The position assumed by Bernays in this regard shows that what Posy considers to be the 'ancient gap between the notion of number and that of a continuous magnitude' is not merely ancient at all. In fact, modern authors still view the discrete and the continuous as unique aspects of the mathematical field of investigation. Rucker (1982) explicitly says: 'The discrete and continuous represent fundamentally different aspects of the mathematical universe' (Rucker, 1982:243). Bell relates this distinction to the entire development of mathematics: 'from the earliest times two opposing tendencies, sometimes helping one another, have governed the whole involved development of mathematics. Roughly these are the discrete and the continuous' (Bell, 1965:12). In their standard work, *Foundations of Set Theory*, Fraenkel, Bar-Hillel, Levy and Van Dalen (1973) hold that the gap between discreteness and continuity is not only central to the foundations of mathematics, but also represents the oldest problem in this (and related) fields: 'Bridging the gap between these two heterogeneous domains is not only the central but also the oldest problem in the foundations of mathematics' (Fraenkel et al., 1973: 212). On the previous page we read: 'Bridging the gap between the domains of discreteness and of continuity, or between arithmetic and geometry, is a central, presumably even the central problem of the foundation of mathematics' (Fraenkel et al., 1973:211). This work is still used as authoritative in respect of the axiomatic foundation of set theory. For this reason Maddy frequently calls upon it (see Maddy, 1997, pp. 14, 39-42, 47-48, 50, 52-54, 57-58, 61, 85).

Anticipating some elements of what follows below a brief purview of the history of mathematics underscores these remarks. Initially, Pythagorean mathematics aimed at explaining the universe in terms of the relationships between integers (i.e., fractions). The statement that *everything is number* is characteristic of this orientation, but soon it was confronted with two discoveries. The first discovery concerns the fact that there are incommensurable geometrical entities, such as the diagonal of a given square which could not be measured by an aliquot part of its side. Fraenkel et al. note that this is equivalent to the fact that in modern terms the square root of the number 2 is not a rational number. The other discovery came from Parmenides and his school to which Zeno and his well-known paradoxes belong. These paradoxes contemplated the impossibility to use infinitely small parts in the construction of finite magnitudes (Becker, 1957; Fraenkel et al., 1973:13).

The effect of these discoveries was that Greek mathematics switched to a *spatial* perspective (discussed in more detail below). This switch not only affected Greek mathematics fundamentally (where the theory of numbers became a part of geometry), but also exerted a decisive influence upon the entire space metaphysics of medieval philosophy. After the Renaissance, partly mediated by the analystic geometry of Descartes, the urge towards arithmeticism slowly emerged once more. Whereas Leibniz (1646-1716) still called God the 'great Geometer,' Jacobi (1804-1851) used the phrase: 'God ever arithmetises.' Cauchy developed this tendency further, although it was only drawn towards its full implications by Weierstrass, Cantor and Dedekind – during the last three decades of the 19th century. The preferential foundational position of number in respect of space is seen in the question asked by Frege:

> Is it not the case that the basis of arithmetic is deeper than all our experiential knowledge and even deeper than that of geometry? (Frege, 1884:44).

Aided by his new notion of a set (a multiplicity of *distinct elements* bound together ino a *whole* – see Cantor – 1962:282), Cantor advanced what he believed to be a purely *arithmetical* concept of a point-continuum (1962:192), defined by him as a *perfectly cohering set* (1962:194). But when Russell and Zermelo independently of each other discovered in 1900 that the naive set concept of Cantor entails logical contradictions, different schools of thought emerged in mathematics. Frege was a victim of this discovery and after unsuccessful attempts to reform his program by the end of his life he once again reverted to a *geometrical* perspective:

> So an a priori mode of cognition must be involved here. But this cognition does not have to flow from purely logical principles, as I originally assumed. There is the further possibility that it has a geometrical source. ... The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis – a geometrical one in fact – so that mathematics in its entirety is really geometry (Frege, 1979:277).

More recently some mathematicians continued emphasizing the primary status of the 'continuum.' Longo (2001) refers to Thom who believes that continuity precedes discreteness:

> For him, as for many mathematicians of the continuum, 'the Continuum precedes ontologically the discrete', for the latter is merely an 'accident coming out of the continuum background', 'a broken line.' (Longo, 2001:6)

Later on in this article Longo combined Thom's views with those of Leibniz: 'By contrast Leibniz and Thom considers the continuum as the original giving, central to all mathematical construction, while the discrete is only represented as a singularity, as a catastrophe' (Longo 2001:19). This view revived the Greek understanding according to which continuity is the simpler concept. Although Fraenkel et al. discern a tendency towards arithmetization underlying Zenon's paradoxes, they point out that the

> converse direction is also conceivable, for intuition seems to comprehend the continuum at once; mainly for this reason Greek mathematics and philosophy were inclined to consider continuity to be the simpler concept and to contemplate combinatorial concepts and facts from an analytic view (Fraenkel et al., 1973:213)

2. **Forms and figures**

From our early childhood we are confronted with all sorts and kinds of *forms* and *figures* – most of the time exemplified in physical shapes, shapes of different kinds of living entities, sensory images (cf. children noticing the full moon), as well as the multifarious diversity of cultural objects. Initially our experience of lines, surfaces and volumes is therefore intimately connected with our sensory abilities, in particular with our eye-sight. At school one, two or three dimensional figures are drawn on a piece paper or on the black board. But already at this primordial level an apparent tension arises between what we can *see* and what we can *imagine*.

3. **Seeing and imagining**

Seeing is straight-forward. But is what we *see* a perfect example of what we have 'in mind' when we *think* of (or imagine) a line, a circle or a cube? Consider for a moment how Plato handled this problem. He distinguishes *being* from *coming into being* and *passing away* because he wants to account for the durability or persistence of knowledge. For this reason he located *being* in the (supra-sensory) intelligible world of static ontic forms and *coming into being* and *passing away* in the world of becoming, that of appearances. (The Greek word 'on' designates what is, what exists, what therefore participates in being.) Visible things fall within the realm of *genesis* where only belief (*doksa*) is possible (that is *eikasia* of the shadow and mirror images and *pistis* of sensory things). In contrast to what is visible, Plato posits that which is solely conceivable and belongs (as *noeton genos*) to the realm of *ousiai* (ontic forms). Contemplating the forms of which visible things are representations, the (*intelligible*) mathematical knowledge is known as *noesis* (cognition in a stricter sense) or as the *episteme* that can be obtained of the *eidè* (ontic forms). In their combination, *dianoia* and *episteme* are known as *noesis* in a broader sense. The principle making it possible for things to be known and giving reality to them, is the idea of the good (*idea tou agathou*) (Cornford, 1966, pp. 216-221).

It is clear that Plato elevated mathematical knowledge to a supra-sensory realm of *noesis* (cognition) in its stricter sense. The ideal objects within Plato's transcendent world of ontic forms do not suffer from the imperfections of what is observable within the sensory world of becoming. This situation is analogous to the relationship between sets of number and multiplicities assumed to be present within the physical world. Cantor committed himself to the idea that the actual infinite exists in concrete reality. He believed that the set of atoms of bodies within the universe is countable (denumerably infinite), while the 'aether atoms' were supposed to display the 'power of the continuum' (that is, equivalent to the set of real numbers). Meschkowski (1972) remarks:

> But modern physics holds a different view on this matter. It operates with the finiteness of the universe. The aether theories are given up and the number of atoms are accepted as finite. … In relation to physical realities there is not a single justification to speak of the set of all natural numbers. (Meschkowski, 1972:345)

Meschkowski also mentions that Cantor expressed this view in a letter to Mittag-Leffler. Meschkowski published a part of this letter, which includes Cantor's reference to the fact that according to him 'the totality of aether atoms has the second power' (Meschkowski, 1967:248). By contrast Hilbert holds: For the number of things which could be an object of our experience, is, even when it is large, beneath a finite limit (Hilbert, 1913:244).

Clearly the modern mathematical theory of real numbers or the mathematical theory of transfinite sets (Cantor) shows that human understanding can grasp abstract structures for which no counter-part is found within (physical) nature. Bernays also points out that our empirical world is restricted to what is (factually) finite. Therefore, according to him, we can only transcend this limitation by 'the formal abstraction which helps us to transcend the boundaries of factuality.' The infinite divisibility of (mathematical) space shows that when this division reaches a sufficient degree of minuteness 'our spatial-metrical representations become meaningless in a physical sense' (Bernays, 1976:38).

This split is similar to the one mentioned above regarding Plato's views. In modern mathematics the weakest Platonistic assumption appears to be given in the acceptance of the totality of integers (see Bernays, 1976:63). With reference to the modern theories of analysis and set theory and their fruitful application within the domains of algebra and topology, Bernays holds that it is not an over-statement to say that the application of Platonism in mathematics is so widespread that it is reigning in mathematics (Bernays, 1976:65).

4. Idealizing and imagining

The influence of mathematical Platonism is also evident in some of the definitions of this discipline. While Meschkowski opts for a definition of mathematics as the science of formal structures, Bernays prefers to characterize it as the science of idealized structures. Yet he views 'idealization' as playing a mediating role between concept and intuition (Bernays, 1976:196).

When we read in Euclid's *Elements* how he defines a point and a line, the above-mentioned distinctions ought to be kept in mind. His first definition states that 'a point is that which has no part' and his second definition claims that 'a line is breadthless length.'

Claiming that a line has no width or breadth idealizes something that cannot be *observed* by the senses, just as little as a point without parts gives access to sensory perception. If no sensorily perceptible physical line is a breadthless length, then humans must have the capacity to imagine an ideal line displaying this property. Wilder observes that 'in a subject like mathematics' the 'conceptual has gradually gained primacy over the observable' (Wilder, 1968:viii).

Human imagination appears to display two qualities. Immanuel Kant says that the imagination is the ability to represent an object in our intuition without its actual presence (Kant, 1787-B:151). But this conception does not sufficiently account for the difference between thought and experience – even what is merely represented in its absence may still reflect the imperfect images of sensory objects (and figures).

It is only when Kant's understanding of the imagination is turned around that this difference clearly surfaces, for it also belongs to the capacity of the human fantasy to imagine that which is given to the senses in a way that is different from how it is sensorily given. In an account of the earliest human artifacts (tools) the archaeologist Narr explains that human tools are distinguished by the fact that they embody the free, formative fantasy of humans, at least when such tools conform to the three conditions identified by him, namely that the form of the manufactured tool must not be given, that the way of production must not be suggested and that the function should not be suggested (such as mere extensions of bodily organs) (see Narr, 1988:281).

The so-called 'ideal' representation of a point and a line is therefore nothing but an instance of the capacity of human beings to imagine something that is given to the senses in a different way (similar to an oval shaped stone which is chopped until it has a sharp end, useful for digging). Yet this imaginative capacity does not stand on its own, because humans are also capable of identifying and distinguishing (lifting out and disregarding), which is normally designated as our capacity to analyze (identify and distinguish) and to abstract (lifting out and disregarding).

5. Fractions and the absence of a primal yardstick

The restriction to what we currently designate as rational numbers (fractions in Greek mathematics) contained unforeseeable problems. One way to explain these problems is to consider the possibility of choosing a measure (yardstick), such that any distance could merely be seen as a multiple of this minimal yardstick. The problem is that the side and diagonal of a square cannot be measured with the same yardstick. Laugwitz explains this by pointing out that there is no 'atomic' primal yardstick (arch-yardstick; *Urmaß*) for all geometrically constructible distances. The side and diagonal of a square (or of a regular pentagram) turned out to be 'incommensurable' (Laugwitz, 1986:14). Of course this was a crisis for the Pythagorean conviction that the essence of everything could be expressed by means of the relationship between integers, that is, in terms of fractions. At the same time it highlights that their arithmeticism (everything is number) demonstrated the underlying abstraction required to articulate their peculiar stance in terms of what is arithmetical (*arithmoi*) – lifting out the arithmetical by disregarding the non-arithmetical aspects. (We have briefly explained above how an abacus can facilitate modal abstraction.) The effect of incommensurability, however, caused a change in the direction of Greek mathematics, away from its initial move towards the arithmetical, because it subsequently explored a geometrical path. Fowler expresses his own preference regarding the outcome of this when he states that this 'first characteristic of early Greek mathematics is negative: it seems to be completely non-arithmetized' (Fowler, 1999:10). Nonetheless he points out that 'one of the main preoccupations of Euclid's geometry is the transformation, combination, and comparison of' figures while the idea 'behind Euclid's use of 'equality' within geometry is one of size, not shape' (Fowler, 1999:12). The problem is nevertheless that size still calls forth the arithmetical (*arithmoi*) – one, two or three dimensional sizes (magnitudes) are all specified by providing numbers, albeit that numerical considerations were now positioned within geometry.

Although it is possible to represent every numerical relationship in a geometrical way, the restriction to integers and their relationships made it impossible to represent every spatial distance arithmetically. Laugwitz remarks:

> This serves as the foundation for the primacy of geometry in respect of arithmetic and the implication is found in the works of Euclid: the theory of numbers forms a part of geometry (Laugwitz, 1986:9).

One of the multiple ways in which the arithmetical appears in a geometrical context, is in connection with the relation between a whole and its parts, such as when a numerical sequence is presented as a half, third, quarter, fifth and so on (see Fowler, 1999:14). The conception of a whole with multiple parts gave rise to the possibility of dividing a whole (and each of its parts) endlessly – which is reminiscent of the view on continuity advanced by Aristotle (Aristotle, 2001:14).

6. **A new impetus to the arithmetization of the continuum**

This Aristotelian legacy dominated the history both of philosophy and mathematics up to the 19th century. The introduction of coordinates already enabled Descartes in introducing algebraic equations and in paving the way for a new attempt to arithmetize our understanding of space. The historically significant break-through of this process took shape during the 19th century. In a posthumously published work on paradoxes of the infinite, Bernard Bolzano provided the first building blocks for Cantor's set theory and transfinite arithmetic. In paragraph 38 of this work he highlights the circularity entailed in assuming a (minimal) length (extension) for points. One cannot explain extension in terms of parts ('points') that are already extended (Bolzano, 1851:72). He concedes that a finite set of points does not generate something *extended*.

Bolzano contemplated what the meaning of 'continuous extension' or 'a continuum' is. He declares:

> a continuum is only then present when there is a set of simple objects (of points …) which are situated in such a way that for every arbitrarily small distance its environment will contain at least one element of this set (Bolzano, 1851:73).

He once more mentions the (traditional) objection that extension cannot be resolved into simple points and then states that:

> an infinite set of points could only then … produce extension when the frequently mentioned condition is met, namely that in every sufficiently small environment every point contains a neighbor (Bolzano, 1851:75)

On this basis he concludes on the same page that one must continue to insist 'that every continuum nonetheless ultimately cannot emerge from anything else but points and once again only points.'

By introducing two criteria, namely coherence and perfectness (a Dedekind cut where each point of division is taken twice, as end-point and as starting-point, and infinite divisibility), Cantor explains that the definition given by Bolzano in § 38 of his *Paradoxien des Unendlichen* is incorrect because it highlights only one feature of the continuum, a characteristic which also covers instances of multiple separated continua. In his work on continuity and irrational numbers (1872) Dedekind explored, according to Cantor, the other property of a continuum, *perfectness* (Cantor, 1932:194). Cantor's own definition on this page combines these two elements, because he defines a point-continuum as a 'perfect-cohering set'.

When every point of a set is a limit-point and when all limit-points of the set belong to the set, Cantor calls it perfect (Cantor, 1962:194). As Editor of the *Gesammelte Abhandlungen* of Cantor, Zermelo here points out that 'perfectness' is what Hausdorff calls 'gaplessness' (see also Wieleitner, 1927:42). Cantor refers to his well-known example of a perfect point-set, which is nowhere dense in any interval no matter how small it is, by introducing 'the set of all real numbers which are contained in the equation

$$z = \frac{c_1}{3} + \frac{c_2}{3^2} + \ldots + \frac{c_v}{3^v} + \cdots$$

where the coefficients c_v arbitrarily have to assume the values 0 and 2 and where the sequence can consist both out of a finite or out of an infinite number of members' (Cantor, 1962:207).

Bolzano still adhered to the view that the integers are genuine numbers, although the integers are also designated as magnitudes. This concept of a magnitude embraces more than numbers, for he says that there are many more magnitudes than numbers and then he mentions fractions and the so-called irrational magnitudes, here designated as expressions – but not numbers (Bolzano, 1851:21). This practice, which speaks of numbers as magnitudes, reflects the long-standing effect of the Greek geometrization of mathematics, because when number is treated within a spatial context, magnitudes appear (length, surface and solid). We have noted earlier that when Euclid mentioned equality he had size (and not shape) in mind (compare once more Fowler, 1999:12).

Dedekind exceeded this remnant of the Greek legacy by connecting the expansion of the number concept to what he designates as the four basic operations, addition, multiplication, subtraction, and division (Dedekind, 1969-1872, pp. 5-6). His approach advanced with a clear awareness of the *distinctness* of number and space, because his line of argumentation in 1872 aims at arithmetically following up what is *given* with the points present on a straight line. His second paragraph reads: *Comparing the rational numbers with the points of a straight line* (Dedekind, 1969-1872, pp. 7-8). The third paragraph deals with the continuity of a straight line (Dedekind, 1969-1872:8-11) and the fourth with the 'creation of irrational numbers' (Dedekind, 1969-1872:11-16). These considerations directed him to the 'continuity of the real numbers' (Dedekind, 1969-1872:16-17).

This line of argumentation of Dedekind is echoed in the contemporary practice to speak of the *number line* in the teaching of mathematics.

7. The number line

A different conceptual aid found to be fruitful in the teaching of mathematical skills and an expansion of the number concept of children, is given in what became known as the *number line*. It is, inter alia, employed as an aid in learning more about adding and subtracting. This concept of a number line combines two distinct perspectives: the numerical and the spatial. Similar to the way in which Dedekind proceeded by arithmetically following up what is given on a straight line, the conception of the number line explores the possibility to discern a one-to-one correspondence between different types of number and particular points on the line. One may imagine the unit line, encompassing the closed interval [0,1], or even, as it is sometimes done, a line representing all real numbers continuing endlessly in the positive and negative directions. Decisive in this regard is the above-mentioned standpoint of Bernays rejecting the widespread view that this distinction between a numerical intuition and a spatial intuition concerns time and space, for according to him the proper distinction needed is that between the *discrete* and the *continuous* (Bernays, 1976:81).

Viewed from this angle the idea of the number line merely establishes a correlation between arithmetical and spatial entities. However, given the inclination present in the thought of Weierstrass, Dedekind and Cantor, namely to arithmetize mathematics completely, one may ask if there is still, within this frame of mind, room for an irreducible continuum?

8. Adolf Grünbaum: the continuum as an aggregate of unextended elements

Grünbaum's aim is to 'inquire whether contemporary point-set theory succeeds in avoiding an inconsistency upon resolving positive linear intervals into extensionless point-elements' (Grünbaum, 1952:288). On the next page the two axioms invoked by Zeno (in his mathematical paradoxes of plurality) are stated, assuming that all magnitudes are divided into positive and 'dimensionless' magnitudes:

(1) the sum of an infinite number of equal positive magnitudes of arbitrary smallness must necessarily be infinite.

(2) the sum of any finite or infinite number of 'dimensionless' magnitudes must necessarily be zero (Grünbaum, 1952:289).

In responding to the second element Grünbaum writes:

> For in the second horn of his dilemma, Zeno avers that a line cannot he regarded as an aggregate of points no matter what cardinality we postulate for the aggregate (Grünbaum, 1952:289).

He proceeds by arguing that the 'completely non-metrical character of all the definitions of dimension theory proper will now become evident and will prove the independence of dimensionality from length' (Grünbaum, 1952:293). He specifies that '[E]very non-empty finite or countably infinite point-set on the number axis is 0-dimensional; consequently, the set R of rational real points is 0-dimensional' too (Grünbaum, 1952:294).

Grünbaum proceeds:

1. There is a one-one correspondence between the points of an n-dimensional Euclidean space E^n and a certain real coordinate system (x_i, . . . x_n)
2. If the points x, y have the coordinates x_i, y_i, then there is a real function $d\,(x, y)$ called their (Euclidean) *distance* given by

$$d(x, y) = \{\sum_1^n (x_i - y_i)^2\}^{1/2}$$

He defines closed, open and half-open intervals: closed $(a \leq x \leq b)$; open $(a < x < b)$ and half-open $(a \leq x < b)$ or $(a < x \leq b)$. He then specifies 'the length of a finite interval (a, b) as the non-negative quantity $b - a$' (the symbols '<' and '=' here have a purely ordinal meaning – Grünbaum, 1952:296; see also De Barra, 1974, pp. 2-4). The point Grünbaum wants to bring home is that

> the set-theoretic addition of a single point to an open interval (or to a half-open interval at the open end) has no effect at all on the length of the resulting interval as compared with the length of the original interval. (Grünbaum, 1952:296).

In the limiting case where $a = b$ the interval is called 'degenerate.' Here 'the closed interval reduces to a set containing the single point $x = a$, while each of the other three intervals is empty' from which it 'follows that the length of a degenerate interval is zero' (loosely speaking Grünbaum here concludes that 'a single point has 0 length' – Grünbaum, 1952:296).

At this point Grünbaum employs the difference between (d)enumerability and non-denumerability – a distinction which we inherited from an initial proof advanced by Cantor in 1874 and which he later on, in 1889, casted in the form of his well-known diagonal proof for the non-denumerability of the real numbers (see the extensive discussion in Strauss, 2011). Whereas the integers and fractions are enumerable – capable of being correlated one-to-one with the natural numbers 1, 2, 3, …, it appeared to be impossible to count all the real numbers in this way. Grünbaum points out that

> the length of an interval which is subdivided into an enumerable number of subintervals without common points is equal to the arithmetic sum of the lengths of these subintervals. Thus, both for a finite number and for a countably infinite number of non-overlapping subintervals, the length L(*i*) of the total interval is an additive function of the interval *i* (Grünbaum, 1952:298).

In connection with Plato's theory of knowledge we noted earlier that his transcendent ideal ontic forms do not suffer from the imperfections of the sensory world of becoming and we compared this situation with the relationship between sets of number and multiplicities assumed to be present within the physical world. Grünbaum distinguishes between the infinite divisibility of a continuous whole (totality) and Cantor's postulate of 'an actual infinity of point-elements *ab initio*' (Grünbaum, 1952:299). Hilbert describes the potential infinite as being in a process of becoming, such as the limit concept in analysis where the infinite appears as the infinitely small and infinitely large [dem Unendlichkleinen und dem Unendlichgroßen], and the actual infinite as a fixed unity (an infinite totality), such as the totality of the numbers 1, 2, 3, 4, … (Hilbert, 1925:167). It is intuitively clearer to designate these two kinds of infinity as the successive infinite (SI) and the at once infinite (AI).

Yet, regarding the question whether or not a line is a continuum of points one first of all has to specify in which sense the term 'line' is understood. Grünbaum therefore states:

> No clear meaning can be assigned to the 'division' of a line unless we specify whether we understand by 'line' an entity like a sensed 'continuous' chalk mark on the blackboard or the very differently continuous line of Cantor's theory. (Grünbaum, 1952:300)

The assumption is that a line which is open to perception by the senses is different from an 'ideal line' *a la* Plato or Cantor.

In the Cantorian view a line-stretch is captured in set-theoretic terms by equating the (actually infinite set of) real numbers with the continuity of a line-stretch. Grünbaum now explores his aim to argue for an understanding of the extended linear continuum as an aggregate of unextended elements. He proceeds from the infinite divisibility of the line and the actual infinite *dividedness* of sets (Grünbaum, 1952:300). This enables him to see a 'positive interval as the union of a continuum of degenerate intervals' combined with the fact 'that the number of lengths to be 'added' is not denumerable.'

The contrast between determining the length of the union of a denumerable infinity of non-overlapping intervals and the degenerate subintervals is that the latter 'are the only kind of (non-overlapping) subintervals of which there are non-denumerably many in a given interval.' But note that Grünbaum had to state that no meaning is assigned to 'forming the arithmetic sum' when an attempt is made to 'sum' a 'super-denumerable infinity of individual numbers (lengths)' (Grünbaum, 1952:301).

Grünbaum proceeds with the attempt to subdivide an interval into a non-denumerable infinity of such intervals and establishes that they cannot be non-degenerate, for Cantor

> has shown that any collection of positive non-overlapping intervals on a line is at most denumerably infinite. It follows that the degenerate subintervals which are at the focus of our interest, are the only kind of (non-overlapping) subintervals of which there are non-denumerably many in a given interval (Grünbaum, 1952:301).

Non-denumerability eliminates the possibility of addition, because what cannot be enumerated cannot be added. Therefore in his theory we cannot meaningfully determine the length of the finite interval (a, b) as the 'union of a continuum of degenerate subintervals' which is achieved by 'adding' the 'individual zero lengths of the degenerate subintervals.' Grünbaum explains that this is an instance which allows for set-theoretic addition while arithmetic addition is not possible (Grünbaum, 1952:302).

The two possibilities discussed by Grünbaum depend upon a choice in favour of one of the two above-mentioned kinds of infinity. If the successive infinite (potential infinite – SI) is chosen, then the length (measure) of an enumerably infinite point-set, upon denumeration of the set, is zero (like the set of rational points between and including 0 and 1). In this case an aggregate of unextended elements cannot constitute the extended linear continuum. What has been shown, according to Grünbaum, is

> that geometrical theory as here presented does not have the paradoxical feature of both assigning the non-zero length $b - a$ to the interval (a, b) and permitting the inference that (a, b) must have zero length on the grounds that its points each have zero length. (Grünbaum, 1952:302).

What has been affirmed are the following four propositions:

1. The finite interval (a, b) is the union of a continuum of degenerate subintervals.
2. The length of each degenerate (sub)interval is 0.
3. The length of the interval (a, b) is given by the number $b - a$.
4. The length of an interval is not a function of its cardinality.

Grünbaum believes that this line of argumentation does not reveal any inconsistency 'against set-theoretical geometry' (Grünbaum, 1952:302). A countable set of degenerate intervals does not constitute an extended linear continuum, because the addition of denumerably many zeros yields nothing but zero. The only way to avoid a 'zero-outcome,' is to consider the non-denumerable case, because then addition is not defined, leaving room open for the jump from zero to a positive (non-zero) value of the interval $b - a$.

The crux of the entire argument therefore rests upon the non-denumerability of the real numbers. Grünbaum is fully aware of the necessity of this assumption: 'The consistency of the metrical analysis which I have given depends crucially on the non-denumerability of the infinite point-sets constituting the intervals on the line' (Grünbaum, 1952:302).

The fundamental question is whether non-denumerability, as *the* presupposition of Grünbaum's argumentation, could be envisaged without an (albeit implicit) appeal to the irreducible meaning of space? From a historical perspective simultaneity was always conceived in connection with space. Just recall how Parmenides described being in spatial terms by affirming that everything coheres in the present (Diels-Kranz, 1959:235). A speculative side-line introduced *eternity* as the timeless present – Plotinus, 1969 Part III, Chapter 7). This legacy is continued by Boethius and more recently by Kierkegaard (the eternal present: *nunc aeternum*). Leibniz juxtaposes time – as 'an order of successions,' with space – as 'an order of coexistences' (Leibniz, 1965:199) and Kant distinguishes three modes of time, namely 'persistence, succession and simultaneity' (Kant, 1787:219).

Every spatially extended figure, such as a line-stretch, can only exist as a whole which is present at once. The term whole derives from the Greek word *holon* and has as Germanic equivalent the term *Ganzheit*. This feature of spatial extension may also be designated with the Latin term *totum* (totality). Although our basic understanding of spatial continuity (extension) entails its infinite divisibility, in the sense of the successive infinite, a deepened perspective may contemplate the actual infinite dividedness of a straight line, as correctly emphasised by Grünbaum (Grünbaum, 1952:300).

There is a strict correlation between simultaneity (at once) and wholeness or totality, entailing that if either of the two is irreducible the other would be as well. That the *whole-parts relation* is indeed unique, is acknowledged by Russell: 'The relation of whole and part is, it would seem, an indefinable and ultimate relation' (Russell, 1956:138). Weyl refers to the understanding of Brouwer in respect of the intimate link between the whole-parts relation and continuity: 'Not in the relation of element and set, but in that of the part to the whole Brouwer observes, in harmony with intuition, the essence of the continuum' (Weyl, 1966:74). Laugwitz argues that the legacy of Cantorian set theory has resolved the continuum into a set of (isolated) points, upon which it then super-imposes, with the auxilliary set theoretical construction of environments and open sets, a 'topology' within which it is once again possible to speak of 'continuity' (see Laugwitz, 1997:266). Weyl arrives at a similar assessment when he affirms that 'it belongs to the essence of the continuum that everyone of its parts allows unlimited further divisions' and 'that the concept of a point must be understood as a limiting idea [*Grenzidee*]' where a ''point' is the representation of the *limit* of a division continued into infinity.' To this he adds a statement similar to the just-mentioned one formulated by Laugwitz: 'In order to recover the continuous coherence of points contemporary analysis, since it has broken the continuum apart into a set of isolated points, had to take recourse to the environment concept' (Weyl, 1921:77).

What is primitive and undefined in Zermelo-Fraenkel set theory makes an appeal to the order of at once as well as to the (spatial) feature of wholeness. In addition to primitive symbols taken from logic, the only set theoretical primitive symbol employed by Zermelo-Fraenkel's set theory is the binary predicate *epsilon*, which denotes the membership relation (cf. Fraenkel et al., 1973, pp. 22-23). This explains why, within Zermelo-Fraenkel set theory, the terms 'set' and 'element' (membership) are synonymous (Fraenkel et al., 1973:24). Frege clarified the confusion of multiplicity and the whole-parts relation by holding that sets *contain* their elements (*members* in Zermelo-Fraenkel set theory) (∈), but *include* themselves and their subsets (⊆) (see Fraenkel et al., 1973, pp. 26-27).

An individual point within the geometrical continuum does not have characteristic properties, which implies that it cannot be distinguished from other points – and the same applies to lines and surfaces. Natural numbers, by contrast, are distinct and possess properties through which they are distinguished from every other number, determined by the numerical order of succession (see Laugwitz, 1986:9). The 'order-place' of every number is determined by this order of succession (captured in what is known as ordinal numbers). Disregarding this order of succession is reflected in cardinal numbers. Fraenkel et al. also articulate the same point:

> Every integer differs from every other in characteristic individual properties comparable to the differences between human beings, while the continuum appears as an amorphous pulp of points which display little individuality (Fraenkel et al., 1973:212).

Of course we have to account for the fact that different number types may imitate crucial spatial features. We may designate these imitations as *analogies* or as *retrocipations* and *anticipations*, i.e., as backward-pointing and forward-pointing analogies (see Figures 1 &2 below). The integers (whole numbers – said to be discrete) imitate the wholeness property of the spatial whole-parts relation. (In set theoretical terms a set is said to be discrete set if it is finite or if it is countably infinite.) The fractions imitate the infinite divisibility of a spatial whole (an extended linear continuum) while the set of rational numbers conform to the conditions for denseness. Finally, the real numbers imitate the continuity of an extended linear continuum (remember that Cantor defined continuity in terms of a perfect-coherent set – see Cantor, 1962:194).

Figure 2 highlights the coherence between the aspects of number and space, viewed from the perspective of the numerical aspect. The idea of an aspect (Figure 1) accounts for the uniqueness of an aspect (its primitive and indefinable core meaning or meaning-nucleus), for its order side (law side – which includes its aspect-specific time order), is factual side, as well as its inter-connections with other aspects – either backward-pointing analogies (retrocipations) or forward-pointing analogies (anticipations). All these structural features are stamped or qualified by die meaning-nucleus of an aspect.

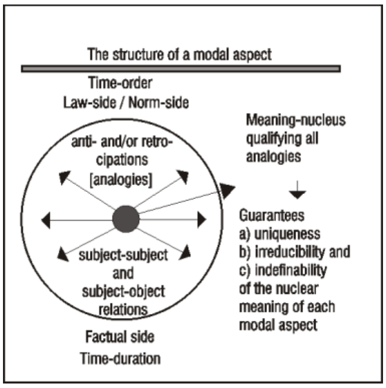

FIGURE 1: The structure of an aspect

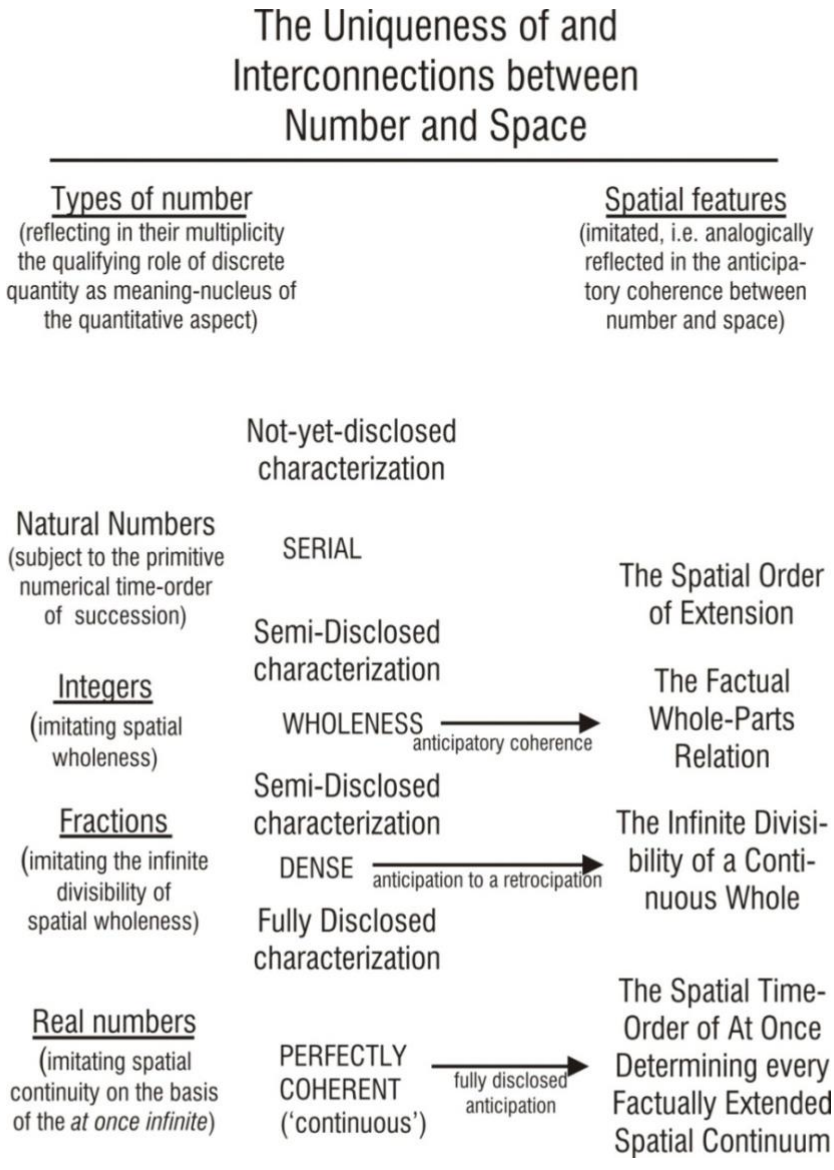

FIGURE 2: The coherence between number and space viewed from the numerical aspect

These 'imitations' represent analogical linkages between the numerical and spatial aspects. Some of them may be pointing backward (retrocipatory analogies, such as the infinite divisibility of a spatial continuum which refers back to the arithmetical order of succession) and other forward (such as the set-theoretic idea of a multiplicity of elements bound together into a whole – Cantor). Speaking of the number of dimensions reflect a numerical analogy on the order side (law side) of the spatial aspect – a dimension is an order of extension and its correlate (at the factual side of the spatial aspect) is also specified by a number (length as a one-dimensional magnitude, surface as a two-dimensional magnitude, and so on).

In general an analogy is present when in the moment of similarity the difference is shown. Both mathematical space and physical space are *extended* (similarity), but whereas the former is both continuous and infinitely divisible the latter is neither continuous nor infinitely divisible (difference). Therefore, within the physical aspect a retrocipatory analogy referring back to the spatial aspect is found. Against this background we can also explain the meaning of the phrase ‘semi-disclosed’ found in Figure 2. The integral nature of integers anticipates the spatial feature of wholeness, but since this feature of wholeness entails the infinite divisibility of a continuous whole the anticipation is mirrored back to the order of (infinite) succession on the law side of the numerical aspect. Therefore, it can be said that the fractions (rational numbers) should be appreciated as an *anticipation* to a *retrocipation* because pointing towards (i.e., anticipating) *wholeness* is reflected in its infinite divisibility which immediately points back to the meaning of number. Intuitionism remains stuck in this *semi-disclosed* meaning of number (see Brouwer, 1952). It is only when *any* succession of numbers is viewed as an *infinite totality*, present at once, that the *full disclosure* or *deepening* of the meaning of number is encountered.

Howevcer, the terminology used in Figure 2 demonstrates that mathematicians, in capturing the imitated inter-connections between number and space, display a sound intuition of the nature of and inter-connections between number and space. The only exception derives from the arithmeticistic aim to reduce continuity to discreteness, focused on a particular understanding of the real numbers. It was not realized that in their imitation of continuity every real number remains *strictly distinct*. When the term *discrete* is used in a restricted sense, merely referring to the nature of the integers, it may best be replaced by an expression such as ‘being integral’. The reason for this suggestion is not only that every natural number, every integer and every fraction is unique, with distinct properties (see once more Fraenkel et al., 1973:212), but that also every real number is unique (distinct). Laugwitz, in his frequently quoted work on *Infinitesimalmathematik*, is therefore fully justified to call upon Cantor’s definition of a set as every combination of a multiplicity of properly distinguished elements into a whole (Cantor, 1966:282), in support of his remark that the *discrete rules* (Laugwitz, 1986:10). He is therefore also correct in remarking that from the outset the set concept was constructed in such a way ‘that what is continuous withdraws itself from its grasp’ (Laugwitz, 1986:10).

Clearly, already the set concept as such entails an element ‘borrowed’ from space, namely the feature of *wholeness* or being a *totality*. Particularly intuitionistic mathematics acknowledges the whole-parts relation as an essential ingredient of the continuum (see Weyl, 1921:77 and Weyl, 1966:74). Intuitionism, however, restricts itself to the successive infinite (SI – the potential infinite) divisibility of a continuum, because it rejects the at once infinite (AI – actual infinity). Only when the numerical meaning of the successive infinite (the SI), is related to the original (and irreducible) spatial meaning of a whole or totality, is it possible to envisage the idea of *infinite totalities*. Cantor simply incorporated the spatial whole-parts relation in his circumscription of a set (‘combining into a whole’) – in set theory expressed in the distinction between *sets* (wholes) and their subsets (their parts).

Bernays holds that the idea of the continuum originally is a geometrical idea, expressed by analysis in an arithmetical language – and he also underscores the fact that the totality character of continuity obstructs every attempt to arrive at a *complete arithmetization of the continuum* (Bernays, 1976:74, 187-190).

In his discussion of the role of an infinite totality in mathematics Wilder refers to the one-to-one correspondence between the natural numbers and the squares which are seen ‘as forming a *complete* infinite totality’ (Wilder, 1973:115). A few pages further he explains it in more detail:

> Following initial work by Cauchy, the efforts of Dedekind, Weierstrass, Cantor, and others were directed intensely towards providing a rigorous theory of real numbers. Their work involved the assumption of a complete infinite totality of real numbers and employed various approaches, such as ‘Dedekind cut’ classes of rational numbers, equivalence classes

of certain sequences of rational numbers ('Cauchy sequences'), and the like (Wilder, 1973:119).

Postulating 'the assumption of a complete infinite totality of real numbers' demonstrates the dependence of this theory of the real numbers upon the irreducible totality character of the geometrical idea of the continuum. Since Cantor's proof of the non-denumerability of the real numbers is fully dependent upon the employment of the at once infinite, articulated in the form of infinite totalities, this implies that non-denumerability stands and falls with the acceptance of the at once infinite. But because the idea of the at once infinite, underlying the idea of infinite totalities, presupposes the spatial feature of wholeness, which is given at once, this form of infinity, while presupposing the irreducibility of spatial continuity, cannot be used in the attempt to reduce spatial continuity to the set of real numbers (or a set of non-denumerable degenerate intervals). Such a set at most (arithmetically) imitates the feature of continuity characteristic of the geometrical idea of the continuum.

Since employing the at once infinite (in proving non-denumerability) presupposes the irreducible spatial order of simultaneity (at once) as well as the irreducible feature of spatial wholeness, it is contradictory to claim that the extended linear continuum could be seen as an aggregate of unextended elements. Although Grünbaum (1952) acknowledged, as mentioned above, that the 'consistency of the metrical analysis which' he has given 'depends crucially on the non-denumerability of the infinite point-sets constituting the intervals on the line' (p. 302), he is patently not aware of the fact that the proof of non-denumerability presupposes the at once infinite which, in turn, presupposes the irreducible spatial order of at once as well as the irreducibility of the spatial feature of wholeness (totality).

Our conclusion therefore is that Grünbaum's attempt to generate a consistent conception of the extended linear continuum as an aggregate of unextended elements failed. Since the extension of a line is one dimensional in nature one can also say that it is an extended *subject* in one dimension. Points, with dimension zero, are *objects* in one dimension (without extension), delimiting a straight line. Hilbert accepts *point*, *straight line*, and *plane* as primitive terms in his axiomatic foundation of geometry (Hilbert, 1913:22; Klein, 1939:160). From a philosophical perspective it can be said that a spatial point (as object) is dependent upon the existence of a spatial subject (such as a continuously extended line-stretch). Therefore it is impossible to account for the spatial extension of a line (as a spatial subject) in terms of spatial objects (points). Points as spatial objects are always *dependent* upon extended spatial subjects.

Euclid's interest in size, namely when equality in geometry is considered, highlights the coherence between extension and magnitude. Length is a one dimensional magnitude, which is specified by a number. This number is a measure of extension and not the extension itself. Therefore Hilbert is justified in avoiding the mistaken view of a straight line, defined as the shortest distance between two points when, in the fourth of his famous 23 problems presented to the international mathematics conference held in Paris in 1900, he spoke of a line as the shortest *connection* between two points (Hilbert, 1970, pp. 302-304). Mac Lane, for example, still says: 'The straight line is the shortest distance between two points' (Mac Lane, 1986:17).

The abstract ('idealized') structure and inter-connections between number and space belong to a dimension of reality inaccessible to sense perception (already appreciated by Plato in his own way) – and this includes the idea of infinite totalities. Bernays correctly points out that we do not have a proper visual representation of infinite totalities. He makes a plea for acknowledging the diversity in which the field of investigation of mathematics differentiates on the one hand, and mutual relationships on the other, including relationships such as that between concrete and idealised structures (Bernays, 1976:188). On the same page Bernays radically rejects, as an arbitrary thesis, what he calls the *arithmetizing monism in mathematics*, because concepts such as 'a continuous curve and plane' and 'in particular those which are disclosed within topology' cannot 'be reduced to representations of number.' He here categorically holds that the continuum, which originally is a geometric idea, cannot be arithmetised exhaustively.

Summarizing the shortcomings in Grünbaum's argument:

1) The 'entire argument [of Grünbaum] is dependent upon the non-denumerability of the real numbers' (conceded by Grünbaum himself);
2) Proving the non-denumerability of the real numbers presupposes the use of the 'at once infinite' (the actual infinite);
3) The idea of *infinite totalities* underlies the idea of the 'at once infinite.'
4) The spatial feature of *wholeness/totality* makes possible the idea of *infinite totalities*.
5) Finally, since the argument aims at reducing spatial continuity to an aggregate of unextended elements by implicitly *using* the (irreducible) spatial feature of wholeness/totality, Grünbaum's argument is self-defeating (contradictory).

9. Concluding summary: what a line really is

Distinguishing between the domains of discreteness and continuity finds support in the history of mathematics and suggests that apart from acknowledging the uniqueness of the aspects of number and space, their unbreakable mutual coherence must be taken into account as well. The meaning of number comes to expression in its coherence with space and likewise the meaning of space comes to expression in its coherence with number (see Figure 1). Bernays mentions the fact that according to Gonseth the whole field of investigation of mathematics differentiates into differently natured 'horizons' while at the same time being related to each other (Bernays, 1976:188).

A straight line or line-stretch instantiates this mutual coherence straightaway. Just recall the above-mentioned (still widely accepted) mistaken view of Mac Lane, namely that '[T]he straight line is the shortest distance between two points.' Also recall Euclid's definition: 'a *line* is breadthless length.' The continuous extension of a line-stretch is not identical to the measure of its extension. Saying that the distance between two points is 3 inches merely specifies an *analogy* of the meaning of number within the spatial aspect. Its infinite divisibility embodies another way in which the meaning of space points backward to the meaning of number. Within space the infinite divisibility of a line analogically reflects the successive infinite in its original (non-analogical) numerical meaning.

Subject to the spatial order of simultaneity a line-stretch is therefore a *one-dimensional factual spatial figure* delimited by two points. Points are not extended and therefore they are not spatial subjects (like lines, planes and volumes) but *spatial objects*, dependent upon the (f)actual extension of a line. As a factual spatial subject, a line is continuously extended, while its length, i.e., its one-dimensional magnitude, is the measure of its extension. If we follow Kant in his (above-mentioned) distinction between succession and simultaneity as modes of time, we may qualify the order-sides of number and space by referring to the numerical *time*-order of succession and the spatial *time*-order of simultaneity (at once).

The spatial subject[line]-object[point] relation, embodied in a (delimited) line-stretch, presupposes the uniqueness and irreducibility of the totality character of continuity (Bernays, Brouwer and Weyl) as well as the irreducibility of the spatial time-order of at once – and at the same time it highlights the mutual coherence between the aspects of number and space.

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