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CONTOURS OF THE HISTORY OF MATHEMATICS: DIVERGING MATHEMATICAL SCHOOLS OF THOUGHT

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ABSTRACT
Although many people appreciate mathematics as a discipline exhibiting sound reasoning with a universal appeal, the mere fact that this academic discipline has a *history* shows that there must have been different and even clashing schools of thought present in it. Greek mathematics advanced under the Pythagorean flag of an "Arithmetica universalis" but soon switched to a spatial perspective owing to the discovery of irrational numbers as an effect of this first foundational crisis of mathematics. The second crisis centred in the *limit concept* which prompted mathematicians to employ the actual infinite. However, this move gave rise to the third foundational crisis when Russell and Zermelo discovered that Cantor's set concept entailed antinomies. This caused the rise of the intuitionistic mathematics of Brouwer and his followers as well as the logicism of Frege, Dedekind and Russell. At once this brought Greek and medieval conceptions to the fore, such as the infinite divisibility of a continuum. The dominant understanding of infinity restricted its meaning to what Kant called the "*successive infinite*" (the potential infinite) – which can never be understood as an infinite totality given at once, i.e., as the *at once infinite* (actual infinite). However, restricting mathematics to the *successive infinite* gave rise to intuitionistic mathematics which arrived at results that find, according to Beth, "no counterpart in classical mathematics". Hilbert followed Kant in accepting certain extralogical concrete objects which are intuited as directly experienced prior to all thinking. But although he accepts the *successive infinite* with Kant, Hilbert still wants to inhabit the paradise created for us by Cantor. At the background of Kant's thought the tension between nature and freedom lurks, also surfacing in the distinction between "Ding-an-sich" and *appearance*. Interestingly, Hilbert made an appeal to Kant's conception of transcendental ideas (concepts of reason) in order to justify his employment of the actual infinite (*at once infinite*). But he did not realize that Kant's view of transcendental ideas entails the idea of infinite totalities. Hilbert had to use the *at once infinite* in order to justify the use of the *at once infinite*. The final irony of his firm belief that it would be possible to prove the consistency of mathematics, is found in the famous 1931 article of Gödel on this issue. In 1946 Hermann Weyl remarked, with a ring to the successive infinite as embedded in human intuition: "It must have been hard on Hilbert, the axiomatist, to acknowledge that the insight of consistency is rather to be attained by intuitive reasoning which is based on evidence and not on axioms." Gödel has shown that a formal mathematical system is consistent or complete. Roos pointed out that Hilbert never recovered from this blow.

1. MATHEMATICS AS "AN EXACT SCIENCE"?

Mathematics appears to be an instance of sound and exact reasoning. Fern alleges that mathematical calculations are "paradigmatic instances of universally accessible, rationally compelling argument." As examples he chooses the familiar argument that 2+2=4. Anyone who "fails to see 'two plus two equals four' denies the Pythagorean Theorem, or dismisses as

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nonsense the esoterics of infinitesimal calculus forfeits the crown of rationality" (Fern, 2002:96-97). Does this mean that all mathematicians agreed about everything within their discipline? Of course not, since the mere fact that mathematics has a *history* points to a negative answer.

2. THE HISTORY OF MATHEMATICS

Therefore, before we can take the leap (with Fern) from Pythagoras to today, we should contemplate the general form of the question just asked: if rationality reigns uncontested within the discipline of mathematics – and in all the other academic disciplines, how do we then explain the *history* of all these disciplines? The "most exact" of them all, namely mathematics, even witnessed three *foundational crises*. Just compare the account given by Fraenkel et.al. (1973:12-14).

History always entails *changes* and if these changes were not *historically significant*, then mathematics would not have had a *history*. A similar claim holds true for all the other academic disciplines, including special sciences such as physics, biology, psychology, logic, the science of history and so on. Keep in mind that more than 100 years ago the newly invented *set theory* revealed nasty antinomies, reminiscent of what occurred in ancient Greece where the discovery of irrational numbers led to the first foundational crisis of mathematics.

3. "ARITHMETICA UNIVERSALIS"

Greek mathematics commenced with the idea of an "Arithmetica universalis" – the Pythagorean idea that *everything is number*. Greek mathematics in the first place was arithmetic and not geometry. They succeeded in arithmetizing the musical intervals and in addition realized that the discovery of harmonious intervals led to the insight that the length of these intervals is related to each other as simple *integers*, i.e., as *fractions* (see Hasse and Scholz 1928:5).

Yet, it was this relationship that confronted Greek mathematics with so-called *incommensurable* magnitudes, leading to the discovery of *irrational* numbers, such as the square root of the number two [$\sqrt{2}$]. Hasse and Scholz remark that this state of affairs was designated by employing the Greek word "alogos" [*ἄλογος*]. If not full-out present this term was accompanied by the awareness of something "widersinnig" (with a contradictory meaning).

4. "IRRATIONAL NUMBERS – THE LIMITS OF ARITHMETIZATION

In the fifth century B.C. two complicated discoveries were made. First of all that "the diagonal of a given square could not be measured by an aliquot part of its side (in modern terms, that the square root of 2 is not a rational number)" and secondly, the paradoxes of Zeno (Achilles and the tortoise, the bisecting paradox, the flying arrow and so on). These paradoxes underscored the apparent impossibility to construe finite magnitudes with the aid of infinitely small parts (Fraenkel et.al., 1973:13-14).

This crisis caused a fundamental switch – away from arithmetic and towards a spatial orientation, which dominated the scene up to Descartes and the early 19th century when it was once more attempted to arithmetize mathematics.

5. THE SECOND FOUNDATIONAL CRISIS OF MATHEMATICS

The hall-mark of the second foundational crisis is found in the circular way in which the limit concept was employed. It manifested itself in the view that irrational numbers (eventually designated as real numbers) could be seen as generated by converging sequences of fractions (still defended in 1821 by the French mathematician Cauchy). Eventually, during the second part of the 19^{th} century, Weierstrass realized that this view is circular, for whatever serves as the limit of a converging sequence of numbers must already be a number to begin with. Weierstrass, Dedeking and Cantor explored an alternative option by postulating, for example, that the $\sqrt{2}$ is the totality of fractions smaller than $\sqrt{2}$, that a so-called Dedekind *cut* should be used in defining real numbers, or that they are fundamental series (Cantor). These three options are equivalent in that these they represent approaches implicitly accepting the *actual infinite* (see Buckley 2012 – Chapters 3 and 4). [Aristotle introduced the distinction between the *potential infinite* and the *actual infinite* but rejected the latter.]

6. THE SET THEORY OF CANTOR – THE WELL-SPRING OF THE THIRD FOUNDATIONAL CRISIS

With the introduction and development of Cantor's set theory during the last three decades of the 19^{th} century it was believed that these problems could be set aside. But unfortunately, Bertrand Russell and Ernst Zermelo (round about 1900) discovered that the set of all sets not containing themselves as elements yields the contradictory situation that such a set is an element of itself if and only if it is not an element of itself.

The third foundational crisis caused some serious problems. In particular the difference between the potential infinite and the actual infinite largely inspired the break-away of intuitionistic mathematics from the main-stream developments. By introducing appropriate axioms, it seemed possible to prevent the deduction of the just-mentioned set of all sets. David Hilbert with awe admires Cantor's theory of transfinite numbers, which is "the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity" (Hilbert 1925:167). Three pages further on he states with equal esteem that no one will drive us out of the paradise created by Cantor.

7. BROUWER – LEAVING THE "PARADISE" CREATED BY CANTOR

Nonetheless the Dutch mathematician, L.E.J. Brouwer, rejected the mathematical use of the actual infinite and with it abolished the transfinite arithmetic of Cantor. This follows from the fact that intuitionism identifies mathematical existence with constructability (see Becker 1973:21 ff.). In his footsteps Kaufmann attempts to reduce the irrational numbers and fractions to integers in his work on the elimination of the infinite from mathematics (Kaufmann 1968:106 ff.). He warns that in speaking about a sequence or succession no *infinite totality* is meant (Kaufmann 1968:122-123).

The gifted mathematician, Hermann Weyl, left the school of Cantor and Hilbert and became an adherent of Brouwer's intuitionism. Another intuitionist mathematician, Arend Heyting, categorically declares that Cantor's theory of transfinite numbers is nothing but a *phantasm* (Heyting 1949:4).

8. MATHEMATICAL REASON UNDER SIEGE

This sheds a shadow over the above-mentioned unqualified trust in "mathematical reason." Morris Kline's assessment of the situation of 20th century mathematics clearly digested this situation:

> The developments in the foundations of mathematics since 1900 are bewildering, and the present state of mathematics is anomalous and deplorable. The light of truth no longer illuminates the road to follow. In place of the unique, universally admired and universally accepted body of mathematics whose proofs, though sometimes requiring emendation, were regarded as the acme of sound reasoning, we now have conflicting approaches to mathematics. Beyond the logicist, intuitionist, and formalist bases, the approach through set theory alone gives many options. Some divergent and even conflicting positions are possible even within the other schools. Thus, the constructivist movement within the intuitionist philosophy has many splinter groups. Within formalism there are choices to be made about what principles of metamathematics may be employed. Non-standard analysis, though not a doctrine of any one school, permits an alternative approach to analysis which may also lead to conflicting views. At the very least what was considered to be illogical and to be banished is now accepted by some schools as logically sound (Kline 1980:275-276).

The decisive insight of Aristotle was that infinity just like continuity only has a *potential* existence which is always *uncompleted*. This basic Aristotelian conception, rejecting *actual infinite* multiplicities, remained the unchallenged general conviction of all mathematicians (if not also all philosophers), until the second half of the nineteenth century when Georg Cantor developed an alternative view in his theory of transfinite arithmetic.[1] In passing we may note that intuitionism constructed an entirely new mathematics (see Kleene 1952:52), while the logician, E.W. Beth, underscores the fact that intuitionistic mathematics replaces these methods by other ones that lead to results which find no counterpart in classical mathematics" (1965:89).

Hermann Weyl became an adherent of the intuitionism of Brouwer. He faced the fact that we have to reconsider the basic insights already operative in Greek philosophy and mathematics. The first important development in Greek philosophy occurred in the school of Parmenides where Zeno discovered the whole-parts relationship (B Fragment 3). This insight turned the most basic understanding of the infinite, manifest in the natural numbers (1, 2, 3, 4, . . .), *inwards*: Whereas the natural succession of numbers is in a literal sense without an end, i.e., endless, a given continuum could be divided without an end. Aristotle holds that if a continuum is divisible, then "it must be divisible either into indivisibles or into divisibles that are infinitely divisible" (Aristotle 2001:317; 231b14-15). He applies this also to "the infinitely divisible line" (Aristotle 2001:715; 994b19). This view of Aristotle was eventually followed by intuitionism (Brouwer and Weyl) by emphasizing that the "essence" of the continuum is not found in the

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1 "Die entscheidende Erkenntnis des Aristoteles war, dass Unendlichkeit wie Kontinuität nur in der Potenz existieren, also keine eigentliche Aktualität besitzen und daher stets unvollendet bleiben. Bis auf Georg Cantor, der in der 2.Hälfte des 19.Jahrhunderts dieser These mit seiner Mengenlehre entgegentrat, in der er aktual unendliche Mannigfalgtigkeiten betrachtete, ist die aristotelische Grundkonzeption von Unendlichkeit und Kontinuität das niemals angefochtene Gemeingut aller Mathematiker (wenn auch nicht aller Philosophen) geblieben" (Becker 1964:69).

relation between an element and set, but between the whole and its parts (1966:74 ff. and 1921:77 – see Strauss 2018).

Anaximander designated the *Archē* (origin) as the *apeiron*, the unbounded-infinite, but even after Anaxagoras de-divinised matter, the infinite still carried with it a *negative* connotation. The remarkable element in this development is that when God was at stake infinity was depreciated and seen as a threat to the (form-giving) *Reason* (Nous). The assumption was that if God was infinite he would not be able to engage in self-contemplation *a la* Aristotle. Eventually Plotinus and Gregory of Nyssa introduced a positive appreciation of God's infinity as part of his perfection. Mühlenberg observes that this view is in conflict with the Greek understanding which saw in infinity the opposite of perfection ["den dort ist die Unendlichkeit das Gegenteil der Volkommenheit" (Mühlenberg 1966:126 note 1)]. Gregory of Nyssa views infinity as an essential characteristic of God. This infinity cannot be traversed (Mühlenberg 1966:166). Nonetheless, contra any form of a negative theology, Gregory of Nyssa affirms that in respect of the Creator of the world we know that he exists and yet do not lie if we hold that we do not know the structure of His essence ["So auch hinsichtlich des Schöpfers der Welt; wir wissen, daβ er ist. Aber wir leugnen nicht, das wir die Struktur seines Wesens nicht kennen" (Mühlenberg 1966:196)].

Regarding the concept of a set Weyl writes: "Not in the relation of element to a set, but in that of a part to the whole Brouwer observes, in accordance with intuition, the essence of the continuum (Weyl 1966:74). Already in 1921 Weyl also pointed out that it belongs to the "essence of the continuum that every part of it allows for an infinite divisibility" (Weyl 1921:77).

These views continue the long-standing restriction of infinity to the potential infinite. Augustin explains in his *Civitas Dei* (the City of God) that although "the infinite series of numbers cannot be numbered, this infinity of numbers is not outside the comprehension of him [God] "whose understanding cannot be numbered." And so, if what is comprehended in knowledge is bounded within the embrace of that knowledge, and thus is finite, it must follow that every infinity is, in a way we cannot express, made finite to God" (Augustin, *The Civitas Dei*, Book XII, Chapter 19). Cantor calls upon this Chapter to show that Augustin accepted the set of natural numbers in its "totality," and therefore as an actually infinite *whole* ("Ganzes" – as a *Transfinitum*) (Cantor 1932:402).

9. AN INFINITE TOTALITY?

Paul Lorenzen argues that within arithmetic no motive could be found for the introduction of the *at once infinite* (Lorenzen 1972:159). Gauss, the Prince of mathematics, protested against the use of the infinite as something completed. In a letter to Schumacher in 1831 he emphatically stated: "So I protest against the use of an infinite magnitude as something completed, which is never allowed in mathematics." ["So protestierte ich gegen den gebrauch einer unendlichen Gröβe als einder volendeten, welches in der Mathematik niemals erlaubt ist" (Meschkowski 1972:31).] In his *Kritik der reinen Vernunft* (CPR), Immanuel Kant acknowledges the infinite divisibility of continuity (Kant 1787:206) and in one instance employs the expression "*successive infinite*" (*sukzessivunendlich*) (Kant 1787:552). To this he adds that the successively infinite series of divisions is never complete (*Ganz*) and therefore does not constitute an *infinite set* ["Da dieser Regressus nun unendlich ist, so sind zwar alle Glieder (Teile), zu denen er gelangt, in dem angegebenen Ganzen als Aggregate enthalten, aber nicht die ganze reihe der Teilung, welche sukzessivunendlich und niemals ganz ist, folglich keine unendliche Menge, und keine Zusammennehmung derselben in einem Ganzen darstellen kann" (Kant 1787:552)].

Yet, when Cantor explored – in the footsteps of Bolzano – the set concept, his basic definition of a set (*Menge*) turned its back on Kant and Gauss: "Under a set we understand every collection *M* of specific properly distinct objects *m* of our intuition or our thinking (which are designated as the 'elements' of *M*) into a whole" ["Unter einer 'Menge' verstehen wir jede Zusammenfassung *M* von bestimmten wohlunterschiedenen Objekten *m* unsrer Anschauung oder unseres Denkens (welche die 'Elemente' von *M* genannt werden) zu einem Ganzen" (Cantor 1932:282)].

The decisive point is that according to Cantor a set combines a multiplicity into a *completed whole*. His understanding of the potential and the actual infinite is intimately linked to this classical definition of a set, for the latter contains the two just-mentioned key features of (a) a multiplicity and (b) a multiplicity combined into a whole (*Ganzheit*). In the case of (a) the potential infinite (*successive infinite*) is seen as a variable finite magnitude which can grow beyond all finite limits or decrease beyond all finite limits [I. Das P.-U. 1 wird vorzugsweise dort ausgesagt, wo eine unbestimmte, *veränderliche endliche* Größe vorkommt, die entweder über alle endlichen Grenzen hinaus wächst … oder unter jede endliche Grenze der Kleinheit abnimmt …;]. In the case of (b) an actual infinite Quantum (the *at once infinite*) in turn is not variable but firm and determined in all its parts, a genuine constant, which at once exceeds every finite magnitude, such as the totality or collection of all finite positive integers which is larger than any finite number. [II. Unter einem A.-U.2 ist dagegen ein Quantum zu verstehen, das einerseits *nicht veränderlich,* sondern vielmehr in allen seinen Teilen fest und bestimmt, eine richtige *Konstante* ist, zugleich aber andrerseits *jede endliche Größe* derselben Art an Größe übertrüft. Als Beispiel führe ich die Gesamtheit, den Inbegriff *aller* endlichen ganzen positiven Zahlen an; diese Menge ist *ein Ding für sich* und bildet, ganz abgesehen von der natürlichen Folge der dazu gehörigen Zahlen, ein in allen Teilen festes, bestimmtes Quantum, …, das offenbar größer zu nennen ist als jede endliche Anzahl.]

This provides the starting-point for a proper understanding of the whole-parts relation: in the finite case a part is not equal to the whole (the decisive classical criterion is that no one-to-one mapping is possible between a part and the whole), whereas in the case of infinite totalities such a one-one mapping is possible ["Hierin besteht das klassische durchschlagende Kriterium der Endlichkeit: Der Teil ist nie gleich dem Ganzen. Erst bei unendlichen Gesamtheiten ist die umkehrbar eindeutige Abbildung des Ganzen auf einen Teil möglich," Hilbert 1992:6)].

10. THE SCIENCE OF THE INFINITE GENERATED DIFFERENT MATHEMATICAL STANDPOINTS

David Hilbert explains the difference between the potential and actual infinite in terms of Cantor's new understanding as follows:

> Someone who wished to characterize briefly the new conception of the infinite which Cantor introduced might say that in analysis we deal with the infinitely large and the infinitely small only as limiting concepts, as something becoming, happening, i.e., with the *potential infinite.* But this is not the true infinite. We meet the true infinite when we regard the totality of numbers 1, 2, 3, 4, . . . itself as a completed unity, or when we regard the points of an interval as a totality of things

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which exists all at once. This kind of infinity is known as *actual infinity* (Hilbert 1925:167 and Benacerraf et. al., 1964:139).

Cantor's distinction between these two kinds of infinity was made fruitful for mathematics in his set theory. But the above-mentioned antinomies that turned up in his *Mengenlehre* (set theory) caused diverging mathematical views. Ultimately it boils down to a questioning of the way in which Weierstrass, Dedekind and Cantor employed the actual infinite in mathematics.

The troublesome outcome of this development is that it affected key elements of what became known as classical "analysis," i.e., that part of modern mathematics involved in investigating the nature of the real numbers (distinct from treating the natural numbers, integers and fractions).

If the *at once infinite* is rejected, essential parts of analysis are amputated. For example, the Bolzano-Weierstrass theorem cannot be proven without the (implicit) use of the *at once infinite*. It states that every bounded infinite subset of R^p has a cluster point. In his significant 1921 article on the "New foundational crisis of mathematics," Weyl announces that he has decided to give up his own attempt (to resolve the third crisis) in order to follow Brouwer ["So gebe ich also jetzt meinen eigenen Versuch preis und schlieβe mich Brouwer an" (Weyl 1921:56)]. The main opponent of intuitionism was the axiomatic formalism of the school of Hilbert.

Although this move of Weyl entails that he does support Brouwer's rejection of the use of the *at once infinite* in mathematics, he still prefers to characterise mathematics as *the science of the infinite* (Weyl 1966:89 – "sie ist die Wissenschaft vom Unendlichen"; see also Weyl 1932:7). The result of merely accepting the *successive infinite* was that, as Stegmüller remarks, intuitionistic mathematics brings to expression its special character "in a series of theorems that contradict the classical results." For example, "while in classical mathematics only a small part of the real functions are uniformly continuous, in intuitionistic mathematics the principle holds that any function that is definable at all is uniformly continuous" (Stegmüller 1970:331; see also Brouwer 1964:79). The Dutch logician Evert Beth also categorically states that it "is clear that intuitionistic mathematics is not merely that part of classical mathematics which would remain if one removed certain methods not acceptable to the intuitionists. On the contrary, intuitionistic mathematics replaces those methods by other ones that lead to results which find no counterpart in classical mathematics" (Beth, 1965:89).

Brouwer himself declares that "in the theories mentioned, mathematical entities recognized by both parties on each side are found satisfying theorems which for the other school are either false, or senseless, or even in a way contradictory. In particular, theorems holding in intuitionism, but not in classical mathematics, often originate from the circumstance that for mathematical entities belonging to a certain species, the possession of a certain property imposes a special character on their way of development from the basic intuition, and that from this special character of their way of development from the basic intuition, properties ensue which for classical mathematics are false" (Brouwer 1964:79).

Hilbert was increasingly frustrated by the criticism of classical analysis by the intuitionism of Brouwer and his followers. Weyl mentions what Hilbert said about disallowing the principle of the excluded middle in the case of infinity: "Forbidding a mathematician to make use of the principle of the excluded middle … is like forbidding an astronomer his telescope or a boxer to use his fists" (Reid 1970:269-270).

In opposition to the logicism of Frege, Dedekind and Russell, Hilbert holds that a precondition for using logical deduction and carrying out logical operations is that "something must be given in conception, viz., certain extralogical concrete objects which are intuited as directly experienced prior to all thinking" (Hilbert 1925:171). Consequently, mathematics cannot solely be founded on logic. This conviction forms an integral part of his theoretical stance that mathematics has a secured content independent from all logic.

The neo-intuitionism of Brouwer, Weyl, Heyting, Van Dalen, Bar Hillel and Levy actually reverses the logicistic thesis that mathematics *is* logic. Heyting remarks that for intuitionism "logic is a part of mathematics and can by no means serve as a foundation for it" since every logical theorem "is but a mathematical theorem of extreme generality" (Heyting 1971:6).

11. KANT AND HILBERT

We have seen that Kant and Hilbert have opposing views of the infinite. The former accepts infinity only in the sense of the "*successive infinite*" whereas Hilbert in addition also accepts the idea of the *at once infinite*. But in terms of the acceptance of extralogical concrete objects which are intuited as directly experienced prior to all thinking, Hilbert actually called upon Kant.

Initially Kant rejects the idea of "a completed infinity," of an *infinite totality* or *an infinite whole*. According to him, "[T]he true transcendental concept of infinity is, that the successive synthesis of units in measuring a quantum, can never be completed" – to which he adds the footnote: "This quantum contains therefore a multitude (of given units) which is greater than any number; this is the mathematical concept of the infinite" (Kant 1787-B:460). On the next page Kant argues that the extension of an infinite world would be given at once. But in order to think such a manifold as a totality it should be necessary to conceive the possibility of a whole (*Ganzen*) through the successive synthesis of a sequence that cannot be completed. In such a case one cannot contemplate the possibility of a whole through the successive synthesis of the parts: Since this synthesis is a sequence that cannot be completed, one cannot contemplate it as a totality. For the concept of a totality itself is in this instance the representation of a completed synthesis of the parts, and this completion, as well as its concept, is impossible ["Da diese Synthesis nun eine nie zu vollendeten Reihe ausmachen" so "kann man sich nicht vor ihr, und mithim auch nicht durch sie, eine Totalität denken. Denn der Begriff der Totalität selbst ist in diesem Falle die Vorstellung einer vollendete Synthesis der Teile, und diese Vollendung, mithin auch der Begriff derselben, ist unmöglich"].

Kant's avoidance of the idea of an *infinite totality* and his restricted understanding of infinity in terms of the *successive infinite*, takes an important turn in the third main part of his *Critique of Pure Reason* (CPR). In his transcendental aesthetic Kant accounts for time and space as internal and external forms of intuition. In his transcendental analytic he explores his understanding of the (a priori) categories of thought, accompanied by the *cogito* (the "I think"). These two parts intend to account for the possibility of synthetic propositions a priori in mathematics and synthetic propositions a priori in physics. These two parts presuppose the distinction between essence (*Ding an sich*) and *appearances*. This distinction ensures that the categories of understanding are not applied to the "thing-in-itself," for then human freedom

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will be sacrificed owing to the unrestricted validity of causality. The freedom of the human soul is for Kant a *Thing-in-itself*, which explains why he emphatically states: "Then, if appearances are Things-in-themselves, freedom could not be rescued" [Denn sind Erscheinungen Dinge an sich selbst, so ist Freiheit nicht zu retten" (CPR-1787:564).

Already in the *Preface* to the second edition of the CPR Kant explained that although the *Thing-in-itself* is unknowable, it is still possible to *think* it (CPR-1787:xxvii). Clearly, the "thing-in-itself" is not *merely* an *idea.* On the contrary, due to the fact that we cannot *know* the "thing-in-itself," but nevertheless can *think* it, there must be a mode of conceptualization in which we can *think*, albeit as something *unknowable*, the "thing-in-itself." Hartmann points out that this is Kant's *transcendental idea* (Hartmann 1957: 311).

The acquisition knowledge advances in three steps. It commences "with the senses, proceeds from thence to understanding, and ends with reason, beyond which there is no higher faculty to be found in us for elaborating the matter of intuition and bringing it under the highest unity of thought" (B:355, cf. B:730). Yet the *unconditioned is* never to be met in experience, but only in the idea – whenever "*the conditioned is given*, *the entire sum of conditions*, *and consequently the absolutely unconditioned* (through which alone the conditioned has been possible) is also given" (B:436). This means that the transcendental ideas are simply "categories extended to the unconditioned" (B:436). But this applies only to those categories in which the synthesis constitutes a *series of* conditions subordinated to one another. To Kant, therefore, the transcendental ideas serve only for *ascending*, in the series of conditions, to the unconditioned (that is, to principles; B:394). *No constitutive* use of these ideas is allowed, because then we only arrive at pseudo-rational dialectical concepts (the source of which Kant called the antinomies – B:672). The three ideas of the *soul* (thinking nature), the *world* and *God* are all to be used in an "as if" way, i.e., *regulatively* (B:710-714).

Kant explains: "Therefore I understand our currently considered pure concepts of reason as transcendental ideas. They are concepts of pure reason; because they view all experiential knowledge as determined by an absolute totality of conditions" ["Also sind unsere jetzt erwogenen reinen Vernunftbegriffe *trasnzendentale Ideen*. Sie sind Begriffe der reinen Vernunft; denn sie betrachten alles Erfahrungserkenntnis als bestimmt durch eine absolute totalität der Bedingungen" (B:383-384)].

What Kant initially rejected is now reintroduced in his understanding of the transcendental ideas of reason – namely the idea of an *infinite totality*. At the end of his above-mentioned article on the infinite Hilbert makes an appeal to Kant's view of transcendental ideas. In the first place he refers to the shortcomings present in the logicist approach when he writes: "In contrast to the earlier efforts of Frege and Dedekind, we are convinced that certain intuitive concepts and insights are necessary conditions of scientific knowledge, and logic alone is not sufficient." And then he proceeds:

> The role that remains for the infinite to play is solely that of an idea – if one means by an idea, in Kant's terminology, a concept of reason which transcends all experience and which completes the concrete as a totality – that of an idea which we may unhesitatingly trust within the framework erected by our theory (Hilbert 1925:190).

However, in the phrase "which completes the concrete as a totality" the idea of an infinite totality (the *at once infinite*) is present. But this idea reveals the fact that Hilbert's approach begs the question. In order to justify the use of the idea of infinite totalities Hilbert made an appeal to Kant's view of the concepts of reason, i.e., transcendental ideas which presuppose the *at once infinite*. He had to use the *at once infinite* in order to justify the use of the *at once infinite*.

Kant was lurking in the background of Hilbert's thought because we noted that with Kant he accepts the role of intuition which ensures the existence of "certain extralogical concrete objects which are intuited as directly experienced prior to all thinking" (Hilbert 1925:171). In 1927 Brouwer gave a series of lectures in Berlin on his approach, including his rejection of the logical principle of the excluded middle (when infinity is at stake). Ludwig Wittgenstein, Hermann Weyl, Kurt Gödel, Karl Menger, and Rudolph Carnap attended his lectures.

Karl Menger, invited Brouwer to Vienna in 1928 where he presented two papers, one on "Wiskunde, Wetenschap en Taal" [Mathematics, Science, and Language] and another one on "The Structure of the Continuum." During these lectures Menger was sitting two rows behind Wittgenstein and noted that Wittgenstein initially appeared to be slightly shocked, an expression that later on turned into a modest smile of satisfaction. Gödel, who was also present, was enthusiastic, as it appears from the diary notes made by Carnap (cf. Roos 2010:24 note 41). Wittgenstein

In 1930 Hilbert spoke on *Knowing nature and Logic* ["Naturerkennen und Logik"]. He rejected *Ignorabimus*, the idea that we shall not know. He was convinced that we must know and shall know ["Wir müβen wissen, wir werden wissen"]. However, one day earlier Gödel explained his new insights, namely that a formal mathematical system is either inconsistent or incomplete. There are propositions with a correct content, but within the formal system of classical mathematics it cannot be proved (see Roos 2010:27). On the next page Roos remarks: "Hilbert never recovered from this blow." ["Hilbert is de slag niet meer te boven gekomen"]. Weyl characterizes this situation succinctly: "It must have been hard on Hilbert, the axiomatist, to acknowledge that the insight of consistency is rather to be attained by intuitive reasoning which is based on evidence and not on axioms" (Weyl, 1970:269).

Hilbert ultimately struggled with the various parts of Kant's CPR and hoped to be rescued by the transcendental dialectic. But in his proof theory, which had to revert to intuitionistic principles – Brouwer still came out on top.

Perhaps Gödel realized that what is required is both an idea of what is unique (primitive) and how it coheres with other instances of uniqueness. Yourgrau mentions that Gödel "insisted that to know the primitive concepts, one must not only understand their relationships to the other primitives but must grasp them on their own, by a kind of 'intuition' " (Yourgrau 2005:169). On the next page he adds that "the fundamental concepts are primitive and their meaning is not exhausted by their relationships to other concepts." However, this insight points at a radical non-reductionist ontology, which exceeds the confines of this article.

Bibliography

Bartle, R.G. 1964. *The Elements of Real Analysis*. New York: John Wiley & Sons, Inc.

Becker, O. 1964. *Grundlagen der Mathematik in geschichtlicher Entwicklung*, München 1964.

Becker, O. 1973. *Mathematische Existenz*, *Untersuchungen zur Logik und Ontologie mathematischer phänomene*, Second unaltered Edition, Tübingen: Max Niemeyer Verlag (1927).

Benacerraf, P. and Putnam, H. 1964. *Philosophy of Mathematics*, Selected Readings. Oxford: Oxford University Press.

Beth, E.W. 1965. *Mathematical Thought*, New York:

Brouwer, L.E.J. 1964. Consciousness, philosophy, and mathematics. In: Benacerraf, P. and Putnam, H. 1964. *Philosophy of Mathematics*. Oxford: Basil Blackwell (pp.78-84).

Buckley, B. 2012. *The Continuity DEBATE*: *Dedekind*, *Cantor*, *du Bois-Reymond and Pierce on Continuity and Infinitesimals*. Boston: Docent Press.

Cantor, G. 1932. *Gesammelte Abhandlungen Mathematischen und Philosophischen Inhalts*. Hildesheim: Oldenburg Verlag (1932).

Fern, R.L. 2002. *Nature*, *God and Humanity*, Cambridge: University Press.

Fraenkel, A., Bar-Hillel, Y., Levy, A. & Van Dalen, D. 1973. *Foundations of Set Theory*, 2nd revised edition. Amsterdam: North Holland.

Hasse, H. and Scholz, H. 1928. Die Grundlagenkrisis der Grieschischen Mathematik. In: *Kant Studien*, Vol. 33(1-2): 4-34.

Heyting, A. 1949. *Spanningen in de Wiskunde*, Groningen.

Heyting, A. 1971. *Intuitionism*. An Introduction. Amsterdam: North Holland Publishing Company.

Hilbert, D. 1925. Über das Unendliche, In: *Mathematische Annalen*, Vol.95, 1925:161-190.

Hilbert, D. 1970. *Gesammelte Abhandlungen*, Vol.3, Second Edition, Berlin: Verlag Springer.

Hilbert, D. 1992. *Natur und mathematisches Erkennen*. Vorlesungen, gehalten 1919-1920 in Göttingen, Nach der Ausarbeitung von Paul Bernays, Herausgegeben von David E. Rowe. 1992, Basel: Springer AG.

Kaufmann, F. 1968. *Das Unendliche in der Mathematik und seine Ausschaltung*, 2nd impression, Darmstadt.

Kant, I. 1781. *Kritik der reinen Vernunft*, 1st Edition (references to CPR A). Hamburg: Felix Meiner edition (1956).

Kant, I. 1783. *Prolegomena zu einer jeden künftigen Metaphysik die als Wissenschaft wird auftreten können*. Hamburg: Felix Meiner edition (1969).

Kant, I. 1787. *Kritik der reinen Vernunft*, 2nd Edition (references to CPR B). Hamburg: Felix Meiner edition (1956).

Kleene, S.C. 952. *Introduction to Methamathematics*, Amsterdam:

Kline, M. 1980. *Mathematics*, *The Loss of Certainty*. New York: Oxford University Press.

Laugwitz, D. 1986. *Zahlen und Kontinuum*. *Eine Einführung in die Infinitesimalmathematik*. Mannheim: B.I.-Wissenschaftsverlag.

Lorenzen, P. 1972. Methodisches Denken, Das Aktual-Unendliche in der Mathematik. In: Meschkowski, 1972 (pp.157-178).

Meschkowski, H. 1967. *Problemen des Unendlichen*. Braunschweig: Vieweg.

Meschkowski, H. 1972. *Grundlagen der modernen Mathematik*. Darmstadt: Wissenschaftliche Buchgesellschaft.

Meschkowski, H. 1972a. Der Beitrag der Mengenlehre zur Grundlgenforschung. In: Meschkowski 1972 (pp.21-55).

Mühlenberg, E. 1966. *Die Unendlichkeit Gottes bei Gregor von Nyssa*. *Gregors Kritik am Gottesbegrff der klassischen Metaphysik*. Göttingen: Vandenhoeck & Ruprecht.

Reid, C. 1970. *Hilbert*, With an appreciation of Hilbert's mathematical work by Hermann Weyl. New York: George Allen & Unwin.

Roos, C. 2010. *Komt onvermoeide arbeid alles te boven*? Afscheidsrede uitgesproken door prof.dr.ir. C. Roos. Faculteit Elektrotechniek, Wiskunde en Informatica. Delft: Technische Universiteit Delft.

Scholz, H. 1969. *Mathesis Universalis*. Second Edition. Darmstadt: Wissenschaftliche Buchgesellschaft.

Strauss, D.F.M. 2018. Is Infinity Purely Arithmetical in Nature? *PONTE*, *International Journal of Sciences and Research*, 74(5/1) May, DOI: 10.21506/j.ponte.2018.5.

Weyl, H. 1921. Über die neue Grundlagenkrise der Mathematik. *Mathematische Zeitschrift*, 10:39–79.

Weyl, H. 1932. Contribution to *The Open World*. New Haven: Yale University Press.

Weyl, H. 1966. *Philosophie der Mathematik und Naturwissenschaft*. Third Edition. Vienna: R. Oldenburg.

Wittgenstein, L. 1968. *Philosophical Investigations*. (1953), Third Edition 1968. Oxford: Basil Blackwell.

Yourgrau, P. 2005. *A world without time*. *The forgotten legacy of Gödel and Einstein*. London: Penguin Books.