Christianity and Mathematics

Danie Strauss

Abstract The trends discernible within the history of mathematics display a recurring one-sidedness. With an alternative non-reductionist ontology in mind this contribution commences by challenging the assumed objectivity and neutrality of mathematics. It is questioned by the history of mathematics, for in the latter Fraenkel et al. (*Foundations of Set Theory*, 2nd rev. ed., Amsterdam: North-Holland, 1973) distinguish three foundational crises: in ancient Greece with the discovery of incommensurability, after the invention of the calculus by Leibniz and Newton (problems entailed in the concept of a limit), and when it turned out that the idea of infinite totalities, employed to resolve the second foundational crisis, suffered from an inconsistent set concept. An alternative approach is to contemplate the persistent theme of discreteness and continuity further while distinguishing between the successive infinite and the at once infinite. Weierstrass, Dedekind, and Cantor define real numbers in terms of the idea of infinite totalities. Frege reverted to a geometrical source of knowledge while rejecting his own initial logicist position. Some theologians hold the view that infinity is a property of God and that theology therefore should mediate its introduction into mathematics. Avoiding the one-sidedness of arithmeticism (over-emphasizing number) and geometricism (over-emphasizing spatial continuity) will require that both the uniqueness of and mutual coherence between number and space is acknowledged. Two figures capture some of the essential features of such an alternative approach. Mediated by a Christian philosophy and a non-reductionist ontology, Christianity may therefore contribute to the inner development of mathematics.

Keywords Christian philosophy · Mathematics · Non-reductionist ontology · Successive infinite · At once infinite · Modal aspects · Herman Dooyeweerd · Limit concept · Irrational numbers

D. Strauss (✉)
North-West University, Potchefstroom, South Africa
e-mail: dfms@cknet.co.za

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Introduction

Part of the fame of mathematics as a scholarly discipline is that it represents the acme of sound reasoning and a manifestation of what it means to be an "exact science." A brief look at the question what mathematics is as well as a brief account of the history of mathematics demonstrates the contrary. According to Fraenkel et al. (1973) mathematics went through three foundational crises. One of the key issues shared by all three is given in the nature of infinity. Oftentimes reflection on Christianity and mathematics assumes that infinity is an attribute of God introduced in mathematics via the discipline of theology. Interestingly, one of the requirements stipulated by Hersh as a test for every philosophy of mathematics is found in the following question: "Does the infinite exist?" (Hersh 1997, 24).

So reflecting on the meaning of infinity may play a mediating role between theology and mathematics as well as between Christianity and mathematics. But also from the perspective of mathematics as such the notion of infinity appears to be of central importance. David Hilbert states: "From time immemorial, the infinite has stirred men's *emotions* more than any other question. Hardly any other *idea* has stimulated the mind so fruitfully. Yet, no other *concept* needs clarification more than any other question." He holds that the "definitive clarification of the nature of the infinite … is needed for the dignity of the human intellect itself" (Hilbert 1925, 163). Hermann Weyl holds that a concise definition capturing the vital core of mathematics is that it is "the *science of the infinite*."[1]

Admiring the supposed "exactness" of mathematics as a scientific discipline faces two challenges: the one is derived from a number of contemporary assessments and the other from the history of mathematics.

Mathematics: The Acme of Sound Reasoning?

Positivism advanced the philosophical claim that science not only ought to be objective and neutral but must also be free from philosophical presuppositions. Sense data (positive facts) should be the ultimate judge. This view is but one among many other philosophical trends advocating what Dooyeweerd calls the *dogma of the autonomy of theoretical reason*. A recent admirer of the rationality of mathematics writes: "Mathematical calculations are paradigmatic instances of a universally accessible, rationally compelling argument. Anyone who fails to see 'two plus two equals four' denies the Pythagorean Theorem, or dismisses as nonsense the esoterics of infinitesimal calculus forfeits the crown of rationality" (Fern 2002, 96–97).

[1] "Will man zum Schluß ein kurzes Schlagwort, welches den lebendigen Mittelpunkt der Mathematik trifft, so darf man wohl sagen: sie ist die *Wissenschaft des Unendlichen*" (Weyl 1966, 89).

By contrast, Hersh mentions four myths regarding mathematics, namely, (i) its *unity*, (ii) its *universality*, (iii) its *certainty*, and (iv) its *objectivity* (Hersh 1997, 37–38). In this work his argumentation entails that these myths cannot account for the history of mathematics as scientific discipline. For example, the widely accepted assertion that mathematics *is* set theory leads into a cul-de-sac. Hersh asks:

> What does this assumption, that all mathematics is fundamentally set theory, do to Euclid, Archimedes, Newton, Leibniz, and Euler? No one dares to say they were thinking in terms of sets, hundreds of years before the set-theoretic reduction was invented. The only way out (implicit, never explicit) is that their own understanding of what they did must be ignored! We know better than they how to explicate their work! That claim obscures history, and obscures the present, which is rooted in history. (Hersh 1997, 27)

Without an ontic reference point mathematics cannot be defined effectively. Without such a point of reference the history of mathematics becomes meaningless. Hersh is therefore justified in holding that "an adequate philosophy of mathematics must be compatible with the history of mathematics. It should be capable of shedding light on that history" (ibid.).

Consider what a mathematician with a sense of the history of mathematics has to say about the recent history of mathematics. Morris Kline states:

> Developments in the foundations of mathematics since 1900 are bewildering, and the present state of mathematics is anomalous and deplorable. The light of truth no longer illuminates the road to follow. In place of the unique, universally admired and universally accepted body of mathematics whose proofs, though sometimes requiring emendation, were regarded as the acme of sound reasoning, we now have conflicting approaches to mathematics. Beyond the logicist, intuitionist, and formalist bases, the approach through set theory alone gives many options. Some divergent and even conflicting positions are possible even within the other schools. Thus the constructivist movement within the intuitionist philosophy has many splinter groups. Within formalism there are choices to be made about what principles of metamathematics may be employed. Non-standard analysis, though not a doctrine of any one school, permits an alternative approach to analysis which may also lead to conflicting views. At the very least what was considered to be illogical and to be banished is now accepted by some schools as logically sound. (Kline 1980, 275–276)

This proliferation of viewpoints does not merely occur within the *philosophy* of mathematics but also within the discipline itself. It is confirmed by Stegmüller:

> The special character of intuitionistic mathematics is expressed in a series of theorems that contradict the classical results. For instance, while in classical mathematics only a small part of the real functions are uniformly continuous, in intuitionistic mathematics the principle holds that any function that is definable at all is uniformly continuous. (Stegmüller 1970, 331; see also Brouwer 1964, 79)

Beth also highlights this point:

> It is clear that intuitionistic mathematics is not merely that part of classical mathematics which would remain if one removed certain methods not acceptable to the intuitionists. On the contrary, intuitionistic mathematics replaces those methods by other ones that lead to results which find no counterpart in classical mathematics. (Beth 1965, 89)

Such a widely diverging assessment of the status of mathematics as a scholarly discipline has deep historical roots. The question is whether we can discern persistent themes running through its development. In their standard work on the foundations of set theory Fraenkel et al. (1973) identify three foundational crises in the history of mathematics.

The First Two Foundational Crises in the History of Mathematics

They first of all mention that “the development of geometry as a rigorous deductive science” in ancient Greece was accompanied by two discoveries. The first one concerns the fact that geometrical entities lacked mutual commensurability, entailing, for example, that “the diagonal of a given square could not be measured by an aliquot part of its side” (that the square root of 2 is not a rational number—Fraenkel et al. [1973, 13]). This was paradoxical because the Pythagoreans were convinced that “everything is number” (see Reidemeister 1949, 15, 30). While it is possible to represent every numerical relationship in a geometrical way, not every relationship between lines could be represented arithmetically. The result of this switch from number to space as modes of explanation is that in the books of Euclid the theory of numbers appears as a part of geometry (see Laugwitz 1986, 9).

After the discovery of the calculus by Leibniz and Newton scant attention was given to its conceptual foundations, causing the second foundational crisis of mathematics emerging at the beginning of the nineteenth century. The weak point was the concept of a limit.

Struggling with the Mathematical Concept of a Limit

Boyer mentions attempts to avoid the *petitio principii* present in Cauchy’s limit concept, such as those pursued by Heine and Cantor (in 1872—Boyer 1959, 289–290). The new way explored by Weierstrass contemplates the totality of all fractions smaller than $\sqrt{2}$. Fraenkel explains:

> So, for example, the “irrational number” $\sqrt{2}$ appears as the cut of which the upper class is constituted by the totality (*Gesammtheit*) of the positive rational numbers larger than 2. It should be pointed out that this reasoning does not contain a kind of existential proof or calculation method for irrational numbers, but a *definition* of new numbers on the basis of known numbers; every other conception is circular. (Fraenkel 1930, 283)

On the same page Fraenkel also points out that the square root of the number 2 cannot be obtained with the aid of the row of rational numbers 1; 1,4; 1,41; 1,414; 1,41,42, and so on, because $\sqrt{2}$ already has to be defined as a number in order to serve as a limit: “A statement such as that the just-mentioned row of rational numbers have the limit-value of $\sqrt{2}$ is only then meaningful when $\sqrt{2}$ has already been defined” (see also Fraenkel 1930, 293). Cantor and Weierstrass realized that they have to introduce the idea of *infinite totalities* in order to understand the nature of real numbers. Weierstrass simply defines $\sqrt{2}$ as the infinite totality of rational numbers smaller than $\sqrt{2}$.

Owing to the second foundational crisis, modern mathematics since 1872 realized that no real number could be generated by introducing converging sequences of rational numbers, such as viewing $\sqrt{2}$ as a number generated through a converging sequence of rational numbers. Whatever serves as a limit of a converging sequence of rational numbers already had to be a number in the first place—therefore real numbers cannot come into being through such a process of convergence. In his *Textbook of Analysis* (1821) it is clear that the French mathematician Cauchy still wrestles with this issue: "When the successive values assigned to a variable indefinitely approaches a fixed value to the extent that it eventually differs from it as little as one wishes, then this last [fixed value] can be characterized as the limit of all the others."[2]

Cauchy still thought that one can obtain an irrational (real) number with the aid of a convergent series of rational numbers, without recognizing the circularity entailed in this argument (see also Weyl 1919). Since 1872 Cantor and Heine made it clear that the existence of irrational (real) numbers is *presupposed* in the definition of a limit. In 1883 Cantor expressly rejected this circle in the definition of irrational real numbers (Cantor 1962, 187). The eventual description of a limit still found in textbooks today was only given in 1872 by Heine, who was a student of Karl Weierstrass with Cantor (cf. Heine 1872, 178, 182). Compare here the original explanation of Heine. He commences by defining an *elementary sequence*: "Every sequence in which the numbers a_n, with an increasing index n, shrink beneath every specifiable magnitude, is known as an elementary sequence" (Heine 1872, 174). In this first section of his article Heine points out that the word *number* consistently designates a "rational number." A more general number (or numeral) is the sign belonging to a sequence of numbers. He secures the existence of irrational numbers with the aid of the following theorem (*Lehrsatz*):

> When a is a positive, non-square integer, the equation $x^2 - a = 0$ does not have an integer as root and consequently also not a rational root. However, on the left hand side it contains, for specific distinct values of x, opposite signs such that the equation has an irrational root. Through this it has been proven that not all number signs can be reduced to rational numbers, *because there exist also irrational numbers*. (Ibid., 186)

This article contains the modern limit concept. However, later on Cantor pointed out that the ideas contained in this article of Heine were actually derived from him (see Cantor 1962, 186, 385).[3] Employing the at once infinite in defining limits (and real numbers) caused the following remark of Boyer: "In a sense, Weierstrass settles

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[2] "Lorsque les valeurs successivement attribuées à une même variable s'approchent indéfiniment d'une valeur fixe, de manière à finir par en différer aussi peu que l'on voudra, cette dernière est. appelée la *limite* de toutes autres"—quoted in Robinson (1966, 269).

[3] In general, a number l is called the limit of the sequence (x_n) when for an arbitrary rational number $\epsilon > 0$ there exists a natural number n_0 such that $|x_n - l| < \epsilon$ holds for all $n \geq n_0$ (see Heine 1872, 178, 179, and in particular page 184 where a limit is described in slightly different terms). Consider the sequence $n/n + 1$ where $n = 1, 2, 3, \ldots$ (in other words the sequence 1/2, 2/3, 3/4, …). This sequence converges to the limit 1.

the question of the *existence* of a limit of a convergent sequence by making the sequence (really he considers an unordered aggregate) itself the number or limit" (Boyer 1959, 286).

Two Kinds of Infinity

The concept of a converging sequence of rational numbers merely employs the most basic understanding of infinity: one, another one, and so on, indefinitely. Traditionally this form of the infinite is known as the *potential infinite* (preferably to be called the *successive infinite*). However, as we just noted, the limit concept which is employed in the calculus (in mathematical analysis) moves one step further by employing what traditionally was designated as the *actual infinite* (preferably to be labelled as the *at once infinite*). The employment of the at once infinite was driven by the urge to arrive at a complete arithmetization of mathematics. Cantor claims that he had no other choice but to employ the possibly most general concept of a purely arithmetical continuum of points.[4] This step accomplished the complete reversal of the geometrization of mathematics in Greek thought by employing the idea of infinite totalities. It also terminated the long-standing reign of the potential infinite (successive infinite). Becker writes:

> The decisive insight of Aristotle was that infinity just like continuity only exists potentially. They have no genuine actuality and therefore always remain uncompleted. Until Cantor opposed this thesis in the second half of the 19th century with his set theory in which actual infinite multiplicities were contemplated, the Aristotelian basic conception of infinity and continuity remained the unchallenged common legacy of all mathematicians (if not all philosophers).[5]

Nonetheless the acceptance or rejection of the at once infinite continued to separate mathematicians. While rejecting the at once infinite, intuitionism also questions the transfinite number theory of Cantor by describing it as a *phantasm* (see Heyting 1949, 4). In his rejection of the actual infinite Weyl even employs biblical imagery. He believes that "Brouwer opened our eyes and made us see how far classical mathematics, nourished by a belief in the 'absolute' that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence" (Weyl 1946, 9). On the next page he alleges that the precise meaning of the word *finite* (where the members of a finite set are "explicitly exhibited one by one") was removed from its limited origin and misinterpreted as something

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[4] "Somit bleibt mir nichts Anderes übrig, als mit Hilfe der in §9 definierten reellen Zahlbegriffe einen möglichst allgemeinen rein arithmetischen Begriff eines Punktkontinuums zu versuchen" (Cantor 1962, 192).

[5] "Die entscheidende Erkenntnis des Aristoteles war, dass Unendlichkeit wie Kontinuität nur in der Potenz existieren, also keine eigentliche Aktualität besitzen und daher stets unvollendet bleiben. Bis auf Georg Cantor, der in der 2. Hälfte des 19. Jahrhunderts dieser These mit seiner Mengenlehre entgegentrat, in der er aktual unendliche Mannigfaltigkeiten betrachtete, ist die aristotelische Grundkonzeption von Unendlichkeit und Kontinuität das niemals angefochtene Gemeingut aller Mathematiker (wenn auch nicht aller Philosophen) geblieben" (Becker 1964, 69).

"above and prior to all mathematics." It was "without justification" applied "to the mathematics of infinite sets." And then he adds the remark: "This is the Fall and original sin of set theory, for which it is justly punished by the antinomies" (ibid., 10).

From his orientation as an axiomatic formalist, David Hilbert appreciates it as the finest product of the human intellect and proclaimed that nobody will be able to drive us out of the paradise created by Cantor.[6]

The Solution as New Source of Trouble

However, what appeared to be a solution to the second foundational crisis of mathematics, namely the employment of the at once infinite, soon turned out to be a key element in the third foundational crisis of mathematics. In 1900, Russell and Zermelo (see Husserl 1979, xxii, 399ff.) independently discovered that Cantor's naïve set concept is inconsistent. Just consider the set C having as elements all those sets which do not contain themselves as elements. Then C is an element of itself if and only if it is not an element of itself.

In this context the fate of Gottlob Frege's logicism is rather tragic. In the appendix to the second volume of his work on the basic laws of arithmetic he had to concede that Russell's discovery (in 1900) of the antinomous character of Cantor's set theory for some time delayed the publication of this volume (1903) because one of the cornerstones of his approach had been shaken.

Close to the end of his life, in 1924–25, Frege not only reverted to a geometrical source of knowledge, but also explicitly rejected his own initial logicist position. In a sense he completed the circle—analogous to what happened in Greek mathematics after the discovery of irrational numbers. In the case of Greek mathematics this discovery prompted the geometrization of their mathematics, and in the case of Frege the discovery of the untenability of his *Grundlagen* also inspired him to hold that mathematics as a whole actually is *geometry*:

> So an *a priori* mode of cognition must be involved here. But this cognition does not have to flow from purely logical principles, as I originally assumed. There is the further possibility that it has a geometrical source.… The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis—a geometrical one in fact—so that mathematics in its entirety is really geometry. (Frege 1979, 277)

For those who still adhere to the arithmetization of the continuum it is important to account for the idea of infinite totalities. Unfortunately, as noticed by Lorenzen, "arithmetic does not contain any motive for introducing the actual infinite."[7] Heyting points out: "Difficulties arise only where the totality of integers is involved" (Heyting 1971, 14). Yet Weierstrass was convinced that his static view of all real

[6] "Aus dem Paradies, das Cantor uns geschaffen [hat], soll uns niemand vertreiben können" (see Hilbert 1925, 170).

[7] "In der Arithmetik … liegt kein Motiv zur Einführung von Aktual-Unendlichen vor" (Lorenzen 1972, 159).

numbers as an infinite totality (from which anyone can be "picked") proceeds in a *purely* arithmetical way.

It is clear that a systematic account of the two kinds of infinity is required since implicitly they surfaced in all three foundational crises in the history of mathematics. From a theological angle, however, it is sometimes claimed that theology has to explain the use of the term *infinity* because it is an attribute of God.

Theology as Mediator Between Christianity and Mathematics?

When mention is made of God's infinity in a theological context, the (theo-ontological) assumption is that infinity originally (eminently) belongs to God. Whenever the notion of infinity is employed in mathematics it is therefore *derived* from the theological understanding of God's infinity. Since Cusanus it is also customary to say that infinity in the sense of endlessness belongs to mathematics, but that the actual infinite is reserved for God only.

This theological legacy does not start by analyzing the structural interconnections between a fiduciary mode of speech (the "language of faith") and the numerical (quantitative) aspect of reality. As a result, it does not realize that *infinity* is a mathematical notion that can only subsequently be employed theologically. The prevailing theo-ontological circle first lifts infinity from its cosmic "place" ("seated" within the arithmetical aspect and its interrelation with the spatial aspect) by assuming that it originally belongs to God. Once this shift is made, it is alleged that infinity can only be (re-!)introduced within the domain of number by taking it over from theology.

An instance of investigating alleged historical (genetic) relationships while ignoring the structural relationships between (the fields of investigation of) theology and mathematics is present in the thought of Chase.

Are Numerical Relations Intrinsically "Theological" in Nature?

No one can deny that both mathematicians and theologians use numerical terms like the numerals *one*, *two*, and *three*. The underlying philosophical issue is, are these notions originally (that is, in a structural-ontic sense) *numerical* notions that are analogically used within a different (faith) context when theologians say that there is but *one* God, or when they speak of God's "tri-unity"? An investigation of the basic concepts and ideas employed in theological parlance reveals a remarkable coherence between the certitudinal aspect of reality and the different other aspects of reality. They are related by means of moments of similarity that are analogically reflected within the structure of the fiduciary aspect of reality (see the explanation of Dooyeweerd's theory of modal aspects explained by Glas and De Ridder in their introductory chapter).

These connections between different aspects are designated as analogical basic concepts of the scholarly disciplines. The simple succession of one, another one, and so on (indefinitely), provides us with the most basic (primitive) awareness of infinity. When this numerical intuition is deepened by our spatial awareness of at once and of wholeness (totality), an infinite succession of numbers may be considered as if all its elements are given at once, that is, as a "completed totality." Our above-mentioned preference to designate these two forms of the infinite as the successive infinite and the at once infinite follows from the fact that both these notions of infinity have their original seat (location/place) within the numerical aspect of reality.

The primitive meaning of number comes to expression in distinctness (discreteness) and (an order of) succession. It provides the basis for the remark of Weyl that it is the principle of (complete) induction which safeguards mathematics from becoming an enormous tautology (Weyl 1966, 86). Continuity in a spatial sense entails both simultaneity (an order of at once) and the notion of wholeness, of being a totality. When a sequence of numbers points towards these two features the primitive numerical meaning of infinity is deepened towards the idea of infinite totalities.

It is therefore surprising that Chase is asking: "Could infinities such as a completed totality be brought into mathematics without a Christian theological foundation?" (Chase 1996, 209). He continues: "At the very least, some idea of God standing outside of our experience must have been necessary, since apart from God we have no experience of the infinite" (ibid.). Chase also mentions the following fact: "Some Scholastics in the Middle Ages and Cantor in the nineteenth century believed in an actual mathematical infinity, based on God's infinity" (ibid., 209–210).

Having failed to investigate any structural relationship between mathematics and theology, the historical analysis provided by Chase precludes the option of acknowledging infinity in both its forms as truly "mathematical." Once the two forms of infinity are recognized in terms of the interconnections between number and space, one can explore additional numerical analogies within the fiduciary aspect. In other words, instead of supposing that "infinities such as a completed totality" originally is a theological idea that is completely foreign and external to mathematics, one would rather account for the numerical meaning of the successive infinite (one, another one, and so on, indefinitely, endlessly, infinitely). Likewise, accounting for the deepening of this primitive meaning of infinity under the guidance of our intuition of simultaneity will point us towards the idea of the at once infinite.

If we proceed from a structural-genetic perspective as the basis of a historical analysis (something absent in Chase's article), one can rephrase the point he wants to make: theological reflection and speculation about the "infinity" of God indeed paved the way for and promoted the eventual mathematical development of a theory of transfinite numbers (Cantor), but in doing that, theology simply digressed into quasi-mathematical considerations which, in the first place, refer to purely mathematical notions related to the inter-modal coherence between number and space.

Chase actually defends a kind of "negative theology." He does not acknowledge the numerical and spatial source of the "potential" and the "actual" infinite, but argues that these terms are originally theological in nature. In the final analysis they are then sent back to the domain from which they were (implicitly) captured in the first place—in the form of theological notions allegedly fruitful for the further development of modern mathematics. This also explains why Chase does not enter into a discussion of the notion of infinity as it is traditionally employed in Christian theology. At least such an investigation might have taken note of the fact that the Bible nowhere explicitly attributes infinity to God. Theologians traditionally extrapolate God's infinity from his omnipresence and eternity. In contrast, recognizing the (deepened) numerical seat of the notion of infinity should rather start from the assumption that theologians could only use notions of infinity as mathematical analogies in their theological argumentation.

Clearly, in the absence of a truly encompassing Christian philosophy, those sincere Christians who want to establish a link between their Christian faith and what mathematics is all about easily fall prey to the long-standing effect of a circular (scholastic) theo-ontology. Such a theo-ontological view proceeds from a particular ontological conception not necessarily inspired by a biblical perspective on reality. A theo-ontological approach takes something that is created and positions it in the "essence" of God, after which it is then finally copied back to creation.

Theo-ontology

Another example of such a theo-ontological approach is found in the work of James Nickel, who claims that the tension between the one and the many "is resolved and answered in the nature of the ontological trinity, the eternal one and the many.… We can do mathematics *only* because the triune God exists. Only biblical Christianity can *account* for the ability to count" (Nickel 2001, 231).[8] One immediately thinks of the second *hypostase* in the thought of Plotinus, the *Nous*, which is also designated as the one-in-the-many (*hen-polla*). The *Nous* does not display an absolute Unity, for it exhibits a balance between unity and multiplicity, as unity-in-the-multiplicity (cf. Plotinus 1956, VI.2.2.2, VI.7.14.11–12). In both instances something from creation is elevated to become a part of God's nature and then afterwards projected back into creation.

[8] Nickel quotes Vern S. Poythress saying: "In exploring mathematics one is exploring the nature of God's rule over the universe; i.e. one is exploring the nature of God Himself" (see Poythress 1976, 184).

Acknowledging the Order of Creation

Alternatively, a Christian approach to God's creational law-order may commence by investigating the fact that throughout the history of mathematics various points of view are found, as briefly noted earlier. Such an undertaking immediately relativizes the claims of "universal reason" which precludes the possibility of genuine divergent (or, conflicting) views within the so-called exact sciences, such as mathematics and physics. In the philosophical legacy primarily operative within the English-speaking world the term *science* is even restricted to the disciplines known as mathematics and physics. What is needed is our acceptance of the primary (and indefinable) meaning of the numerical aspect (where, as *part of creation*, our awareness of the one and the many finds its seat). Everything is not number, but everything functions within the numerical aspect. This remark presupposes the ontic givenness of the various modal (functional) aspects of reality, including number and space. Existence is more than *entitary* existence. In a different context I have argued that the lifting out of a particular aspect while disregarding others constitutes the distinctive feature of scholarly thinking, to wit, *modal abstraction* (see Strauss 2011b).

From this it follows that a scholarly account of the nature, scope, and limits of any scientific discipline in principle exceeds the limits of such a discipline (special science). The moment a mathematician wants to define or describe what mathematics is all about, something is said that falls *outside* the universe of discourse of mathematics as a discipline. Suppose it is alleged that mathematics is the discipline investigating "formal structures" (Bernays), that it is the "science of order" (Russell), that it is the "science of order in progression" (Hamilton), or even that it is a discipline constituted by the two subdisciplines algebra and topology (Bourbaki). Then we have to realize that in all these (and many more similar) "definitions" something is said *about* mathematics without, in any way, getting involved in *doing mathematics*. It is therefore a simple fact of philosophy of science that talking about mathematics cannot be equated with being involved in doing mathematics. The decisive criterion in defining mathematics is not *who* defines mathematics but the *nature* of such a definition of mathematics. Defining mathematics belongs to the philosophy of mathematics.

The Discrete and the Continuous

We have seen that the history of mathematics toggled between the extremes of an arithmetized and a geometrized perspective and that one of the key issues is found in alternative views of the nature of the infinite. Fraenkel et al. connect this to the relationship between the "discrete and continuous." They even speak about a "gap" in this regard which has remained an "eternal spot of resistance and at the same time of overwhelming scientific importance in mathematics, philosophy, and even physics" (Fraenkel et al. 1973, 213).

Apparently the "discrete admits an easier access to logical analysis," explaining according to them why "the tendency of arithmetization, already underlying Zeno's paradoxes, may be perceived in axiomatic set theory." Yet since "intuition seems to comprehend the continuum at once" the Greeks believed that continuity could be comprehended at once, explaining their inclination "to consider continuity to be the simpler concept" (ibid.). Recently a number of French mathematicians returned to the priority of space (the continuum)—they are even known as mathematicians of the continuum (see Longo 2001).

Yet the mentioned urge to arithmetize mathematics inspired Cantor to employ the general concept of a purely arithmetical continuum of points.[9] This step accomplished the complete reversal of the geometrization of mathematics in Greek thought by employing the idea of infinite totalities.

Unfortunately, modern set theory turned out to be burdened by the troublesome presence of what Cantor called *inkonsistente Vielheiten* (inconsistent sets) (see Cantor 1962, 447). Zermelo introduced his axiomatization of set theory in order to avoid the derivation of "problematic" sets while Hilbert dedicated the greater part of his later mathematical life to develop a proof of the consistency of mathematics. But when Gödel demonstrated that in principle it is not possible to achieve this goal, Hilbert had to revert to intuitionistic methods in his proof theory ("meta-mathematics").

This orientation is still alive within mathematics. We have quoted Fraenkel et al. (1973, 213) regarding the basic position of continuity. More recently, Longo highlights the above-mentioned views of René Thom and other mathematicians: "For him, as for many mathematicians of the continuum, 'the continuum precedes ontologically the discrete,' for the latter is merely an 'accident coming out of the continuum background,' 'a broken line'" (Longo 2001, 6). He also remarks: "By contrast Leibniz and Thom consider the continuum as the original giving, central to all mathematical construction, while the discrete is only represented as a singularity, as a catastrophe" (ibid., 19). Of course Longo is quite aware of the fact that the set theory of Cantor and Dedekind assigns priority to notions of discreteness "and derive[s] the mathematical continuum from the integers" (ibid.; cf. 20).

The history of mathematics therefore opted at least for three different possibilities: (i) to attempt exclusively to use the quantitative aspect of reality as mode of explaining for the whole of mathematics—Pythagoreanism, modern set theory (Cantor, Weierstrass), and axiomatic set theory (axiomatic formalism—Zermelo, Fraenkel, von Neumann, and Ackermann); (ii) to explore the logical mode as point of entry (the logicism of Frege, Dedekind, and Russell); and (iii) to assert the geometrical nature of mathematics—an attempt which was once again taken up by Frege, now close to the end of his life, and by the mentioned mathematicians of the continuum.[10]

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[9] See the quotation given in n. 4 above.

[10] Bernays also consistently defended the position that continuity belongs to the core meaning of space and that the modern approach of Cantor and Weierstrass to mathematical analysis did not accomplish a complete arithmetization of the continuum (see Bernays [1976, 188] and also Strauss [2011a]).

Christianity and Scholarship

The legacy of reformational philosophy, particularly in the thought of Herman Dooyeweerd, proceeds from the basic biblical conviction that within itself created reality does not find an ultimate or final mode of explanation. The moment a thinker attempts to pursue this path, the honour due to God as Creator, Sustainer, and ultimate *Eschaton* of created reality is dedicated to a mere creature. The distorting effect of this inclination is manifest in all the antinomous "isms" discernible within all the disciplines (not only within mathematics). The challenge is to explore the option of acknowledging the uniqueness and irreducibility of every aspect inevitably involved in practicing mathematics without attempting to reduce any of the modal aspects to another. Dooyeweerd claims that whenever this anti-reductionist approach is not followed, theoretical thought inescapably gets entangled in *theoretical antinomies*. His claim is, in addition, that the logical principle of non-contradiction finds its foundation in the more-than-logical (cosmic) principle of the excluded antinomy (*principium exclusae antinomiae*) (see Dooyeweerd 1997, 2:37ff.).

A Christian attitude within the domain of scholarship, while observing the *principium exclusae antinomiae*, will attempt to avoid every instance of a one-sided deification or reification of anything within creation. The biblical perspective that God is Creator and that everything within creation is dependent upon the sustaining power of God opens the way to acknowledging the life-encompassing consequences of the redemptive work of Christ, for in him we are in principle liberated from the sinful inclination to search for a substitute for God within creation. We are in principle liberated from this inclination in order to be able—albeit within this dispensation always in a provisional and fallible way—to respect the creational diversity and its dependence upon God with the required intellectual honesty.

The Meaning of an Aspect Comes to Expression in Its Coherence with Other Aspects

This depth dimension contains the supra-theoretical motivation for articulating a non-reductionist ontology. The dimension of cosmic time and that of modal aspects constitute essential elements in such an ontology. Discerning the uniqueness and coherence within the diversity of various aspects within created reality is guided by the philosophical hypothesis that no single aspect could ever be understood in its isolation from all the other aspects. Furthermore, the core meaning of an aspect is indefinable and this insight entails the indispensability of "primitive terms." Yourgrau explains that Gödel "insisted that to know the primitive concepts, one must not only understand their relationships to the other primitives but must grasp them on their own, by a kind of 'intuition'" (Yourgrau 2005, 169).

The crucial point is that the meaning of an aspect only comes to expression in its unbreakable coherence with other modes—exemplified in what is designated as the *modal analogies* within each modal aspect, reflecting the inter-modal coherence between a specific aspect and the other aspects. These analogies are retrocipatory (backward pointing) or anticipatory (forward pointing) in nature. They are therefore also known as *modal retrocipations* and *modal anticipations*. Within the quantitative aspect of reality no retrocipations are found and within the certitudinal aspect there are no anticipations.

Christianity impacts mathematics through the non-reductionist ontology of a biblically informed philosophy. Such a philosophical orientation aims at avoiding the one-sided conflicting "ismic" trends operative throughout the history of mathematics. What it entails is summarized in the following thesis: accept the *uniqueness* and *irreducibility* of the various aspects of created reality, including the aspects of quantity, space, movement, the physical, the logical-analytical, and the lingual (or sign) mode, while at the same time embarking upon a penetrating, non-reductionist analysis of the inter-modal connections between all these aspects.

This proposal crucially depends upon a more articulated account of the theory of modal aspects as such. Figure 1 captures the most important features of a modal aspect.

Although Christians and non-Christians are living in the same world and do the same things, they indeed do these *differently*. What does this mean for mathematics?

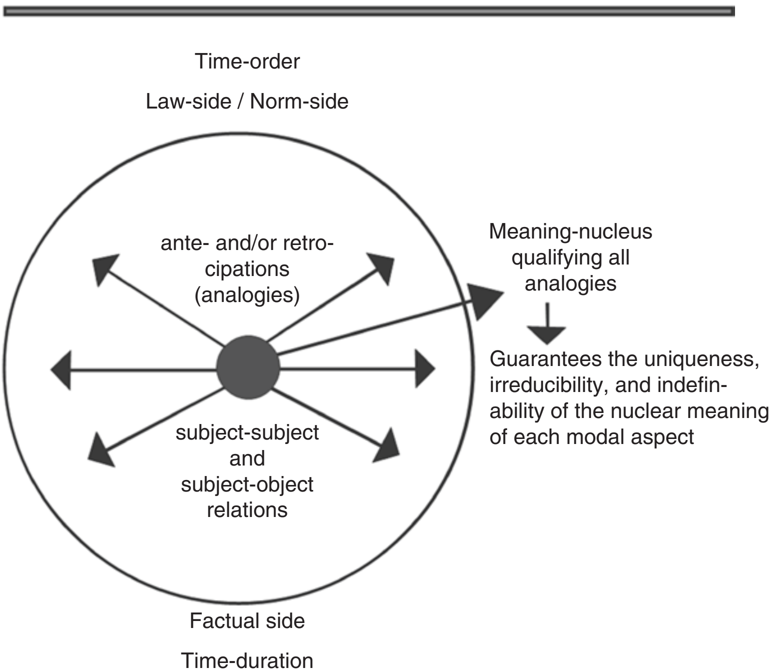

Fig. 1 The structure of a modal aspect

Infinity as Point of Entry

We will restrict ourselves to some general remarks regarding the inter-modal coherence between the aspects of number and space as it comes to expression in the earlier-mentioned two kinds of infinity, namely, the successive infinite and the at once infinite.

Since Descartes advanced the nominalistic conviction that “number and all universals are mere *modes of thought*” (*Principles of Philosophy* 1.57) only concrete entities were considered to be “real.” Yet the question is whether there is not, prior to any human intervention, construction, or cognition, a *given* multiplicity of aspects or functions of reality. Hao Wang points out that Kurt Gödel[11] is very “fond of an observation that he attributes to Bernays”; to wit, “That the flower has five petals is as much part of objective reality as that its color is red” (quoted in Wang 1988, 202).

Surely the quantitative side or aspect of entities is not a product of thought, for at most human reflection can only explore this given (quantitative) trait of reality by analyzing what is entailed in the meaning of multiplicity. Bernays is a prominent mathematician who explicitly questions the dominant view that ascribes reality only to entities. He says that this mistaken conception acknowledges one kind of factuality only, namely that of the “concrete” (Bernays 1976, 122).[12]

Perhaps the most impressive advocate of the ontic status of modal aspects is found in Gödel’s thought. He advances the idea of “semiperceptions” in connection with “mathematical objects.” Next to a physical causal context within which something can be “given,” Gödel refers to data of a second kind, which are open to semiperceptions. Data of this second kind “cannot be associated with actions of certain things upon our sense organs” (quoted in Wang 1988, 304). In terms of a Dooyeweerdian approach these semiperceptions relate to the functional aspects of reality. Gödel says: “It by no means follows, however, [that they] are something purely subjective as Kant says. Rather they, too, may represent ‘*an aspect of objective reality*’ but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality” (ibid.; my italics).[13]

Theoretical and non-theoretical thought can explore the *given* meaning of this quantitative aspect in various ways. It is first of all done by forming *numerals* or *number symbols*, such as “1,” “2,” “3,” and so on. The simplest act of counting

[11] At the young age of 25 Gödel astounded the mathematical world in 1931 by showing that no system of axioms is capable—merely by employing its own axioms—of demonstrating its own consistency (see Gödel 1931). Yourgrau remarks: “Not only was truth not fully representable in a formal theory, consistency, too, could not be formally represented” (Yourgrau 2005, 68). The devastating effect of Gödel’s proof is strikingly captured in the assessment of Hermann Weyl: “It must have been hard on Hilbert, the axiomatist, to acknowledge that the insight of consistency is rather to be attained by *intuitive reasoning* which is based on evidence and not on axioms” (Weyl 1970, 269).

[12] Note that Bernays employs the term *factual* in the sense in which we want to employ it—referring to what is given in reality prior to human cognition.

[13] Kattsoff defends a similar view where he discusses “intellectual objects” which are also characterized by him as “quasi-empirical” in nature (Kattsoff 1973, 33, 40).

already explores the original meaning of the quantitative aspect of reality. It occurs in a twofold way, because every successive number symbol ("1," "2," "3," etc.) is correlated with whatever is counted.[14]

Figure 1 highlights the correlation between the law-side and factual side of an aspect and it also captures the way in which the dimension of cosmic time comes to expression in the first two modal aspects distinguished by Dooyeweerd—by acknowledging the numerical time-order of succession and the spatial time-order of simultaneity.

The Attempt to Arithmetize Continuity

We shall now briefly focus on the attempt to arithmetize continuity by calling upon the non-denumerability of the real numbers. When the elements of a set can be correlated one-to-one with the natural numbers [0, 1, 2, 3, …] then it is called *denumerable* (*countable*). Cantor has shown that the integers and fractions are denumerable. So the question remained, what about the real numbers?

In his famous diagonal proof Cantor only considered the closed interval [0,1] (i.e., $0 \leq x_n \leq 1$) which represents all the real numbers. In his diagonal proof Cantor commences by assuming that there is an enumeration $x_1, \; x_2, \; x_3, \; \ldots$ of real numbers satisfying $0 \leq x \leq 1$ given by

$$x_1 = 0.a_1 a_2 a_3 \ldots$$

$$x_2 = 0.b_1 b_2 b_3 \ldots$$

$$x_3 = 0.c_1 c_2 c_3 \ldots$$

Now another real number is considered, namely, y conforming to the stipulation that y_1 is different from a_1; y_2 from b_2; y_3 from c_3, and so on. This number y certainly also belongs to the closed interval [0, 1] because $0 \leq y \leq 1$.

Nonetheless it differs from every x_n in a least one decimal position, from which Cantor concludes that every attempt to enumerate the real numbers within this interval will leave out at least one real number. The conclusion is that the real numbers are uncountable or non-denumerable. Grünbaum attempted to use this outcome in developing a consistent conception of the extended linear continuum as an aggregate of unextended elements. His entire argument crucially depends upon the non-denumerability of the real numbers and he is fully aware of the necessity of this assumption: "The consistency of the metrical analysis which I have given depends crucially on the non-denumerability of the infinite point-sets constituting the intervals on the line" (Grünbaum 1952, 302).

[14] Frege correctly remarks "that counting itself rests on a one-one correlation, namely between the number-words from 1 to *n* and the objects of the set" (quoted in Dummett 1995, 144).

Cantor starts his diagonal proof with the assumption that all real numbers in the closed interval [0, 1] are arranged in a countable succession. They are given as an infinite totality. He then shows that another real number (*y*) could be specified in such a way that it differs from every real number in their assumed denumerable succession (each time at least in the *n*th decimal place).

The crucial point is that the entire diagonal proof presupposes the assumption of an infinite *totality* containing all its elements *at once* (that is, conforming to the spatial time-order of simultaneity).

Sometimes the idea of an infinite totality is not accepted, such as one finds in intuitionistic mathematics (this school of thought only accepts the successive [potential] infinite). When the at once infinite is rejected, Cantor's proof does not yield non-denumerability, because there is no constructive transition from the successive infinite to the at once infinite (see Heyting 1971, 40; Fraenkel et al. 1973, 256, 272; and Fraenkel 1928, 239n1). Without the idea of an infinite totality Cantor's diagonal proof merely demonstrates that for a given sequence of successively infinite sequences of numbers it will always be possible to find another sequence of numbers differing from the given sequence of successively infinite sequences of numbers at least in one decimal place. In this interpretation non-denumerability nowhere surfaces.

Proceeding from a non-reductionist ontology, no contradiction is at stake, because the two kinds of infinity merely reflect inter-modal connections between number and space. We noted earlier that the primitive meaning of infinity is given in the successive infinite: one, another one, and so on, endlessly, infinitely. Once this primitive meaning of infinity is deepened under the guidance of the spatial time-order of *simultaneity* (at once—on the law-side) and the (factual) spatial *whole-parts relation*, it is possible to employ the idea of the at once infinite. Therefore, Cantor's diagonal proof is not contradictory even though an undisclosed (not-yet-deepened) use of the infinite does not yield non-denumerability. The latter can only be inferred when the at once infinite is assumed (see Strauss [2014] for a more extensive treatment of this issue).

The diagonal proof demonstrates a couple of things that are of central importance for the relationship between Christianity and mathematics. First of all, it shows that an apparently exact mathematical proof may lead to opposing results depending upon the underlying philosophical view on the relationship between number and space. Secondly, it highlights the implications of a non-reductionist ontology for mathematics as an academic discipline by suggesting an alternative for the historical movement to and fro between the aspects of number and space. Instead of reducing space to number or number to space—twice accomplished in the history of mathematics—one should accept their uniqueness (sphere sovereignty) and their mutual coherence (sphere universality).

It should be noted that the rejection of the at once infinite by intuitionistic mathematics resulted in a different theory of the real numbers (normally referred to by mathematicians as *the continuum*) instead of merely imitating a spatial feature in a numerical way—see Fig. 2.

Types of number (imitated, i.e., analogically reflected in the anticipatory coherence between number and space)		Spatial features (reflecting in their multiplicity the qualifying role of discrete quantity as meaning-nucleus of the quantitative aspect)
	Not-yet-disclosed characterization	
Natural numbers (subject to the primitive numerical time-order of succession)	SERIAL	The spatial order of extension
	Semi-disclosed characterization	
Integers (imitating spatial wholeness)	WHOLENESS ⟶ anticipatory coherence	The factual whole-parts relation
	Semi-disclosed characterization	
Fractions (imitating the infinite divisibility of a spatial whole)	DENSE ⟶ anticipation to a retrocipation	The infinite divisibility of a continuous whole
	Fully disclosed characterization	
Real numbers (imitating spatial continuity on the basis of the at once infinite)	PERFECTLY COHERENT ⟶ fully disclosed anticipation ("continuous")	The spatial time-order of at once determining every factually extended spatial continuum

Fig. 2 The uniqueness of and interconnections between number and space

The Intuitionism of Dooyeweerd and Dummett

Interestingly, Dooyeweerd followed intuitionism in its rejection of the at once infinite (see Dooyeweerd 1997, 1:98–99n1, 2:92, 340n1). He did not realize that his own theory of modal aspects provides a point of departure for a unique and meaningful account of the nature of the at once infinite, namely as a regulative hypothesis in which the meaning of the numerical time-order of succession is deepened by anticipating the spatial time-order of at once. This deepened hypothesis makes it possible to view any successively infinite sequence of numbers as if it is given at once, as an infinite whole or an infinite totality.

It is worth mentioning that when Dummett explains the intuitionistic notion of infinity, a striking dialectic surfaces. He both uses and discards notions that are essentially spatial in nature, for without any hesitation he speaks about "infinite *totalities* of mathematical objects" (my italics). On the next page the expression "infinite *domain*" is used as a substitute for "infinite totality" (cf. similar usages—Dummett 1978, 22, 24, 57, 58, 59, 63, and so on). Sometimes the phrase "infinite *structures*" is used (ibid., 56, 62). At the same time Dummett holds on to the successive infinite since according to him any infinite structure is never something given as a set of "completed objects" (cf. Dummett 1978, 62).

Because both the time-order of at once and the whole-parts relation are embedded in the irreducibility of the spatial aspect, it begs the question if an attempt is made to reduce continuity to discreteness on the basis of the non-denumerability of the real numbers since this feature of the real numbers, as we noted, presupposes the at once infinite while the latter presupposes the spatial time-order of at once. The upshot of such an attempted arithmetization is antinomous and comes to expression in the following contradiction: continuity (space) could be reduced to number if and only if it cannot be reduced to number.

Approximating the Theory of Modal Aspects

Although Paul Bernays did not develop a theory of modal aspects, he has a clear understanding of the cul-de-sac entailed in arithmeticistic claims. He writes:

> It should be conceded that the classical foundation of the theory of real numbers by Cantor and Dedekind does not constitute a complete arithmetization.… The arithmetizing monism in mathematics is an arbitrary thesis. The claim that the field of investigation of mathematics purely emerges from the representation of number is not at all shown. Much rather, it is presumably the case that concepts such as a continuous curve and an area, and in particular the concepts used in topology, are not reducible to notions of number [*Zahlvorstellungen*]. (Bernays 1976, 187–188)[15]

Concluding Remark

Exploring an analysis of more interrelations between number and space exceeds the scope of this contribution. However, the preceding discussion sufficiently supports the claim that Christianity—via a Christian philosophy and a non-reductionist ontology—does have a meaningful contribution to make to the discipline of mathematics.

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[15] "Die hier gewonnenen Ergebnisse wird man auch dann würdigen, wenn man nicht der Meinung ist, daß die üblichen Methoden der klassischen Analysis durch andere ersetzt werden sollen. Zuzugeben ist, daß die klassische Begründung der Theorie der reellen Zahlen durch Cantor und Dedekind keine restlose Arithmetisierung bildet. Jedoch, es ist sehr zweifelhaft, ob eine restlose Arithmetisierung der Idee des Kontinuums voll gerecht werden kann. Die Idee des Kontinuums ist, jedenfalls ursprünglich, eine geometrische Idee. Der arithmetisierende Monismus in der Mathematik ist eine willkürliche These. Daß die mathematische Gegenständlichkeit lediglich aus der Zahlenvorstellung erwächst, ist keineswegs erwiesen. Vielmehr lassen sich vermutlich Begriffe wie diejenigen der stetigen Kurve und der Fläche, die ja insbesondere in der Topologie zur Entfaltung kommen, nicht auf die Zahlvorstellungen zurückführen."

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