Frege's Attack on "Abstraction" and his Defense of the "Applicability" of Arithmetic (as Part of Logic)

Daniël F M Strauss

Faculty of the Humanities
P.O. Box 339
Bloemfontein 9300

E-mail: <fgds@rs.uovs.ac.za>

Abstract:
The traditional understanding of *abstraction* operates on the basis of the assumption that only *entities* are subject to thought processes in which particulars are disregarded and commonalities are lifted out (the so-called method of *genus proximum* and *differentia specifica*). On this basis Frege criticized the notion of abstraction and convincingly argued that (this kind of) "entitary-directed" abstraction can never provide us with any *numbers*. However, Frege did not consider the alternative of "property-abstraction." In this article an argument for this alternative kind of abstraction is formulated by introducing a notion of the "modal universality" of the arithmetical and by developing it in terms of the distinction between *type-laws* (laws for entities – applicable to a limited class of entities) and *modal laws* (obtaining for every possible entity without any restriction). In order to substantiate this argument a case is made for the acceptance of an *ontic foundation* for the arithmetical (and other modes or functions of reality – with special reference to Cassirer, Bernays, Gödel and Wang), which, in the final section, serves to give an ontological account of (i) the connections between the arithmetical and other aspects of reality and (ii) the applicabality of arithmetic. In the course of the argument the *impasse* of logicism is briefly highlighted, while a few remarks are made with regard to the logical subject-object relation in connection with Frege's view that number attaches to a concept.

The limitations of typological abstraction

Originating in Greece, the employment of typological abstraction continued far beyond this initial period of Western philosophizing. Its main concern is to account for concretely existing *things* or *entities* within reality. It may be surprising to us that Greek thought apparently found a point of rest in the delimitation provided to their world picture by the large world-sea, the *Okeanos*. The dominant position of Aristotle in Greek philosophy established the classificatory method which disregards *distinctive particulars* in order to arrive at more abstract *genera*. Yet, according to their understanding, the earth is a circular slice *delimited* and *surrounded* by the *Okeanos*. It seems strange that they did not *transcend* the boundaries of the *Okeanos*. Our own modern acquaintance with the idea of *infinity* almost automatically forges this question.

However, in terms of the Greek mind this was impossible. The *Okeanos* was one of the primal forces that were subdued (and brought within delimitation) when the Olympic gods started their reign. The ordered cosmos owes its *form*, *measure*, *harmony* and *determination* (concept) to these gods. Whatever finds itself *outside* this limit does not display any form-delimitation and can therefore not be *conceived*. As a consequence Aristotle does not acknowledge an *abstract* (or *empty*) space. He lacks our modern concept of space. According to the mature Greek understanding *space* does not exist, only *place*. Place is a property exclusively attributed to a *concretely existing body*. In the absence of a body there is no subject for the predicate *place*. From this it naturally follows that an empty place is the *place* of *nothing – in other words, it is no place* at all!

The possibility to understand (and encompass) the ordered cosmos flows from its *finite* and *limited* nature – for that reason science is restricted to this finite, limited and ordered cosmos.

During the Middle Ages the Latin formulation of this (entitary-directed) method of classification became well-known in the form of the distinction between a *genus proximum* and *differentia specifica*. This method of concept-formation is well at home within domains where a *typological* classification is required, such as is the case within *biology* as a discipline.[1]

Although this kind of hierarchical progress does lead to higher levels of *abstraction*, it does not move away from *entitary reality* as such. Being a mammal is not less real than being this or that kind of mammal or even this *individual* mammal (except, once again, if one joins *nominalism* in rejecting all universals outside the human mind).

However, the higher (abstracted) levels reached in such a *typological classification* are still related to and focused upon entities (or at least their *structures*) and cannot serve as a foundation for those kinds of properties of things which come in sight when the *typicality* of entities is *disregarded*.[2]

For example, just think about the (supposed) abstract nature of mathematical thinking. Some mathematicians tend to divorce their subject matter from every possible "application." If some or other part of mathematics turns out to have found an application outside the domain of mathematics those mathematicians with a *platonistic* inclination find it "miraculous." David Hilbert, the foremost mathematician of the 20th century, recalls the statement of Gauss, namely that pure number theory is the *queen of mathematics* and adds the remark that it did not find an application anywhere (Hilbert, 1935:386). Yet, Frege represents a different position in this regard by opposing the distinction between pure and applied mathematics. In the second volume of his

1 The modern neo-Darwinian theory of evolution, given its nominalistic orientation (according to which "organisms are not types and do not have types" – Simpson, 1969:8-9), of course handles a different concept of structure altogether.

2 The (universal) *conditions for* being this or that *type* of thing must be distinguished from the (universal) way in which particular entities evince their conformity with these conditions. In *being an atom* or *being human*, this or that atom / human being shows that it meets the conditions for what it is. The term "structure" is therefore ambiguous. It may refer to the *order for* (*structural law* or *structural principle* for) the existence of a specific type of entities, whereas the *structures of* these latter reveal what is correlated (and therefore *distinct*) from the said *order for* structures of entities. A *structure for* has the meaning of a *law for*, while a *structure of* represents the universal way in which individual entities reveal their conformity with the given law for its existence (also known as its *law-conformity*).

Grundgesetze (1903:§91)[3] he writes: "It is applicability alone that raises arithmetic from the rank of a game to that of a science. Applicability therefore belongs to it of necessity" (translation from Dummett, 1995:60).

Surely the difficulty with this distinction is not simply that different subdivisions of mathematics turned out to serve "applications" in other disciplines, but that this emphasis *constantly changed*. In the foreword to Volume V of their work on the nature and application of mathematics (with reference to computers, algebra and analysis) Behnke et al. remark that what was *abstract* yesterday is now considered to be *concrete* (for example matrices). What yesterday was seen as *pure* is today viewed as *applied* (e.g., functional analysis). What yesterday was under *suspicion* is now seen as *respectable* (e.g., the theory of probability). Areas of mathematical reflection that had been considered *beyond all possibilities of application* later on turned out to be *useful in different ways*.

Dummett explains that it was Frege's objective to destroy the illusion that whenever mathematics finds an application, a *miracle* occurred:

> The possibility of the applications was built into the theory from the outset; its foundations must be so constructed as to display the most general form of those applications, and then particular applications will not appear a miracle (Dummett, 1995:293, cf. 300).

Exploring the meaning of what is here called "the most general form" of a theory can help us to understand the *ontic* basis of seemingly purely conceptual clusters – but not without first considering the argument advanced by Frege to explain that numbers belong to the class of "objects" which are both "non-actual" and "objective." In order to achieve this aim Frege first of all launches a severe attack on the prevalent theories of *abstraction* as defended by some of his prominent contemporaries.

We have mentioned that the traditional (*genus proximum* and *differentia specifica*) method of concept-formation is focused upon concrete entities in reality. Pursuing its path allows for increasingly higher levels of generality. When Frege phrases his objections against the idea of *abstraction* his arguments are restricted to this kind of *entitary-directed abstraction*.

He states that the properties through which entities distinguish themselves from each other are indifferent in respect of their number (1884:40 ff.). He explicitly asks the question from what one should abstract in order to arrive at the number "one" when one starts with the *moon* as an entity. By abstraction, he proceeds with his argument, one only arrives at concepts such as: "attendant of the earth," "attendant of a planet," "celestial body without its own light," "celestial body," "body," "object" (Gegenstand) – and nowhere in this series the number "1" will occur (1884:57; §44).[4] Dummett mentions the example where Frege refers to a white cat and a black cat in order to highlight the shortcomings of "abstraction": "The concept 'cat', which has been attained by abstraction, indeed no longer includes peculiarities of either; but just for that reason, it is a single concept" (Frege, 1884:45-46; §34; translation by Dummett, 1995: 84).

3 In our subsequent analysis references to Frege (1884 and 1893) will simply be to the relevant paragraphs – thus following the practice employed by Dummett in his penetrating and encompassing work of 1995.

4 Note that this example fully fits the requirements of the *genus proximum*/*differentia specifica* method.

Frege is first of all reacting against the prevailing view that *number* is to be seen as a *set of units* – the pure "ones" – obtained, via "abstraction" from the concrete "objects" we can experience. It is exactly at this point where Angelelli thinks that Frege's criticism becomes devastating:

> by abstracting from the particular differences and natures of the given objects no plurality can be attained, but only one thing (the concept 'cat', for example) (Angelelli, 1984:467).

In his remark related to Cantor's definition of a subset (Cantor, 1962:282), Zermelo also refers to the attempt to introduce the notion of "cardinal number" with the aid of a process of abstraction, which would imply that a cardinal number is to be seen as a "set composed of pure ones."[5] If these "ones" are still *mutually distinct* then they simply provide the elements of a newly introduced set equivalent to the first one, which means that the required abstraction did not help us at all (cf. his remark in Cantor, 1962:351).

Where Kant argues for the *synthetic* nature of mathematical judgments in his *Critique of Pure Reason* (CPR), he clearly realizes that *pure logical addition* (a merely *logical synthesis*) cannot give rise to a *new* number (cf. CPR, 1787:15 where he considers the proposition that 7+5=12). In a different way Frege made the same point: entitary-directed abstraction can only proceed to more abstract *entities*, but can never yield any *number* as such. The logical addition of "ones" or "twos" cannot but end with the repeated identification of another number of the same kind: having identified a "two" and another "two" still results only in the "abstract" notion of "twoness."

Number: the road to logical objectivity (Frege)

However, is it possible to distinguish different (modal) *properties* of one and the same *entity*? In terms of Frege's example of the moon, we may be more specific: does the moon have any *numerical* properties? Frege indeed realizes that "number" is an answer to the question "how many?" and explicitly discusses this question in connection with the moon (1884:57; §44). But in the absence of a theory of *ontic functions* (*modalities* – to be discussed below), he cannot relate the numerical properties of entities to the (universal) ontic meaning of the quantitative aspect of reality and categorically denies that "number" is the "property of something" (Frege, 1884:63; §51). This question relates to the fundamental *numerical* question acknowledged by Frege, namely the question: '*How many*'?[6] Is the moon "one" or "more than one"? he asks. Obviously, these questions point in another direction – the direction of what may be called "property-abstraction." This kind of abstraction, surely, is fundamentally different from *entitary-directed abstraction*.

Yet, there is an even more fundamental issue at stake, because the question "How many?" requires a *human response*. Are there (universal) *ontic* features presupposed

5 The *cardinality* or *power* of a set disregards any order-relation between its elements. When such an order-relation is kept in mind, ordinal numbers are at stake. Counting the "first," the "second" and so on therefore employs ordinal numbers. Cantor holds that the concept of a cardinal number emerges when, with the aid of our active thought-capacity, we abstract from the character of the different elements of a given set M and also disregards the order in which they are given (1962:282).

6 In his *Grundgesetze* Frege distinguishes between cardinal numbers (*Anzahlen*) – an answer to the question "How many objects of a certain kind are there"? – and real numbers (numbers employed for *measurements*) (Cf. – Frege, 1903:§157 and Dummett, 1995:64).

in our answer to this question which are *quantitative in nature*? Or, alternatively, do we have to revert to the position that number and all universals are *creations of human thinking*?[7] In 1881 Frege wrote in an article on "Booles rechnende Logik und die Begriffsschrift" (unsuccessfully submitted for publication): "individual things cannot be assumed to be given in their totality, since some of them, such as number for example, are first created by thinking" (quoted by Dummett, 1995:3).[8]

> Frege's primary aim is to develop a *logicist thesis* according to which arithmetic ought to be reducible to logic. The beginning of the main text of *Grundgesetze* posits that the aim of *Grundlagen* was to show that *arithmetic is a branch of logic*. In the *Introduction* to *Grundgesetze* he specifies this aim by saying that "no ground of proof needs to be drawn either from experience or from intuition" (translation by Dummett, 1995:3). On the basis of his definition of an "ancestral" given in *Begriffsschrift* Frege defines natural numbers in *Grundlagen* (cf. §79, §83): natural numbers are "those objects for which finite mathematical induction holds good (Dummett, 1995:12).[9]

Within the realm of "objective ideas" Frege distinguishes between "objects" and "concepts," keeping in mind that the correlate of a *concept* is an *object* (Dummett, 1995:66). Frege's famous distinction between "Sinn" and "Bedeutung" (*sense* and *reference*) is one drawn *within* the realm of the "objective" (Dummett, 1995:12). Whereas the *Bedeutung* (content or meaning) of an expression initially for Frege was at once its *significance* and *what it signified*, his new distinction between *Sinn* and *Bedeutung* subdivided the former notion of *content* into the said two components of *sense* and *reference* (cf. Frege, 1893:x and Dummett, 1995:67).

The proposal of John Stuart Mill that number could be obtained through inductive generalization is ultimately rejected by Frege because he places natural numbers within a domain which is not open to sensory perception. In his significant work on substance and function Cassirer explains the limitations of the psychological conception of presentation (in contrast to the "*logical* meaning of the concept of number") as follows: "The characteristic relations which prevail in the series of numbers are not thinkable as properties of the given contents of presentation. Of a presentation it is meaningless to say that one is larger or smaller than another, the double or triple of it, that one is divisible by another, etc." (Cassirer, 1953:33).

Dummett points out that for Frege the concept *natural number* comprises "all and only those objects attainable from 0 by reiterating the successor operation" (1995: 63).[10]

7 In his *Principles of Philosophy* Descartes says "that number and all universals are only modes of thought" (Part I, LVIII).

8 In connection with the distinction to be drawn between *modal* and *typical* concepts we shall pay attention to the peculiar meaning attached by Frege to the term "quantity."

9 According to Freudenthal, Dedekind was perhaps the first one (cf. Dedekind, 1887, §59, §80) to call the conclusion from *n* to *n+1* complete induction ("vollständige Induktion"). Neither Bernoulli nor Pascal is the founder of this principle. Its discovery must be credited to Francesco Maurolico (1494-1575) (cf. Freundenthal, 1940:17). In a mathematical context, where "bad induction" is supposed to be excluded (as Freundenthal remarks – 1940:37), no adjective is necessary to qualify the term *induction*.

10 Since "succession" finds its original "seat" within the numerical mode of reality this insight actually contradicts the logicistic aim of Frege – similar to the way in which, in general, the axiom of infinity obstructed the attempt to reduce mathematics to logic.

What is intrinsic to the concepts of arithmetic, according to Frege, preventing them from becoming *synthetic* – the mistake of Mill and Helmholtz – is the *general principle governing all possible applications* (Dummett, 1995:60). What is meant here is that when one answers the question "How many?" one needs to be "quite unspecific as to the type of objects concerning which the question 'How many?' could be answered by citing a natural number" (as explained by Dummett, 1995:61). Dummett adds the phrase: "for that reason, it involved no concept peculiar to any non-mathematical subject-matter" (1995:61).[11]

Where Cassirer introduces, in his rejection of "objects" either belonging to inner or outer reality, the alternative option of "acts of apperception" with which "the numerical determination is connected," he points out that even a person who wants to restrict knowledge to what is given in the senses "recognizes this universality" (i.e., in terms of our suggestion what we will call below *modal universality*): "but it understands it, according to its fundamental theory, as a thing-like 'mark,' which is uniformly found in a group of particular objects" (Cassirer, 1953:33-34). He then quotes Mill who says: "All numbers, must be numbers of something: there are no such things as numbers in the abstract. But though numbers must be numbers of something, they may be numbers of anything. Propositions, therefore, concerning numbers, have the remarkable peculiarity that they are propositions concerning all things whatever; all objects, all existences of every kind, known to our experience" (Cassirrer, 1953;33-34 – reference to Mill, *A System of Logic*, Book II, Chapter 6, 2).

Moreover, the phrase which we have used, namely "the numerical aspect of reality," implicitly refers to an *alternative view of reality*, which is foreign to Frege's understanding of the nature of number, because, according to him (in the words of Dummett), "that to which, in general, a number is ascribed is a concept" (1995:74).

When something can act as a causal agent Frege calls it *Wirklich*. Dummett argues convincingly that it should not be taken to mean *real* as opposed to *ideal*, but rather be understood as his "manner of distinguishing between what present-day philosophers usually call 'concrete' and 'abstract' objects" (Dummett, 1995:80); and numbers belong to the class of "objects" which are objective but not *Wirklich* (Dummett, 1995:81).

Although Dummett claims that Frege "brilliantly, decisively and definitively" refuted the abstractionist idea that numbers are to be seen as sets of featureless units (amongst others adhered to by Husserl and Cantor) (1995:82), the implicit assumption continues to be one which (albeit *correct* in itself!) argues that the inherent limitations of entitary-directed abstraction can never produce a concept of number:

> His essential, and crucial, contention in *Grundlagen* was that abstraction is (at best) a means of coming to grasp certain general concepts: as a mental operation, it has no power to create abstract objects or abstract structures (Dummett, 1995:85).

In all of this we have to appreciate the strength of Frege's argument, particularly evidenced in the question whether or not the units of an abstract set of featureless units are identical with or distinct from one another. Frege argues that we "cannot succeed

11 Against the background of our suggested distinction between *entitary-directed abstraction* and *property-abstraction* one may argue in this context that disregarding what is "peculiar to any non-mathematical subject-matter" precisely amounts to what the idea of *property-abstraction* aims at.

in making different things identical simply by operations with concepts; but, if we did, we should no longer have *things*, but only a single thing" (Frege, 1884, §35; Dummett's translation, 1995:86).[12]

As abstract, non-actual logical objects (entities), numbers are determinate objects of scientific enquiry:

> We speak of 'the number one', and indicate by means of the definite article a single, determinate object of scientific enquiry. There are not distinct number ones, but only a single one. In 1 we have a proper name, and, as such, it is as incapable of a plural as 'Frederick the Great' or 'the chemical element gold' ... Only concept-words can form a plural (Frege, 1884:§38; Dummett, 1995:87).

Frege finally gives the following answer to the question what is a number a number of, that is, what is a number ascribed to: "The content of an ascription of number consists in predicating something of a concept."[13] Dummett summarizes this result as follows: "what a number is ascribed to is a concept" (1995:88). Since the notion of a *concept* is prior to that of the *extension* of a concept, a class can be given only as the extension of a concept (Dummett, 1995:92).[14] The question is whether the ascription of a number is not much rather an issue of – *identifying* it – which is different from *ascribing* a number to a concept. However, we shall return to this question in connection with what will be called the logical subject-object relation below.

What is "ontic" about "number"?

Epistemology traditionally is well at home with sensory perception ("sense data" in the positivistic legacy) and with the ontic status of (at least: physical) entities (usually simply referred to as "objects"). Measured against this yardstick it is not surprising that the quantitative nature of number is transposed into the domain of thought products as creations of the human mind. Particularly when it concerns the nature of *infinity* the temptation seems to be to transfer it to the domain of *pure thinking*. David Hilbert (implicitly) still continues this orientation when he argues that after we have established the finiteness of reality in two directions (with regard to the infinitely small and in respect of the infinitely large), it may still be the case that the infinite does have a justified place *within our thinking* (I am italicizing – DS)![15]

However, already the co-worker of Hilbert, Paul Bernays, holds that the treatment of number presupposes the representation of a totality of numbers as a system of things as well as the totality of sets of number – which is not an *arbitrary construction*: "One cannot justifiably object to this axiomatic procedure with the accusation that it is arbi-

12 Frege quotes Jevons who says: "It has often been said that units are units in respect of being perfectly similar to one another; but though they may be perfectly similar in some respects, they must be different in at least one point, otherwise they would be incapable of plurality" (cf. Dummett, 1995:86).

13 Supported by a substantial and convincing argument this is Dummett's translational equivalent of Frege's German statement: "dass die Zahlangabe eine Aussage von einem Begriffe enhalte." (See Frege, 1884:59; §46 and Dummett, 1995:87, 88).

14 As "objects" classes are denoted by singular terms (Dummett, 1995:92), but although numbers attach to the concept of the objects being counted, they are not *properties of concepts* (cf. Dummett, 1995:96, 108).

15 "Die Endlichkeit des Wirklichen haben wir nun in zwei Richtungen festgestellt: nach dem Unendlichkleinen und dem Unendlichgroßen. Dennoch könnte es sehr wohl zutreffen, daß das Unendliche in unserem Denken einen wohlberechtigten Platz hat und die Rolle eines unentbehrlichen Begriffes einnimmt" (1925:165).

trary since in the case of the foundations of systematic arithmetic we are not concerned with an axiom system configured at will for the need of it, but with a systematic extrapolation of elementary number theory conforming to the nature of the matter (*naturgemäß*)."[16] Bernays explicitly questions the dominant conception that only *one kind of factuality* ought to be recognized, namely that of the "concrete" (Bernays, 1976:122).[17] Kattsoff, for example, also makes a plea for the acknowledgement of both physical and mathematical factuality, although "mathematical objects" are "quite different from physical objects": "They are clearly not the sort of things that can be observed by means of the senses" (1973:30). Through intellectual involvement "mathematical objects" come into sight: "In analogy to physical objects which are called sensory objects because they are observed by the senses, mathematical objects may also be called intellectual objects (or rational objects?) because they are observed by the intellect" (1973:33). Later on he calls his approach "quasi-empirical" (1973:40).

It is noticeable that some of the most prominent mathematicians of the 20th century increasingly struggled with the "objective" status of the *subject matter* of mathematics. Of course, the positivistic legacy, combined with the nature of experimental physics, emphasizing *sense perception*, constantly challenged mathematicians to come up with an account that will at least be similar (or: analogous) to the supposed way in which *perception* serve as a gateway to *knowledge of reality*. Yet its is clear to them that "mathematical objects" are not plainly open to sense perception. In other words, in order to account for the subject matter of mathematics there ought to be something "out there" which differs from the "concrete" entities ("objects") we can experience through sense perception. The extreme platonist ought to clarify this issue, because the assumed "objective existence" of "mathematical objects" is in need of some other foundation if it is not merely postulated as *constructions of the human mind*.

The search for something "ontic" in the subject matter of mathematics, however, does not need to exclude a *constructive role* played by the mathematician. For, if we accept for a moment *ontic quantitative properties* which may serve as a starting-point for mathematical reflection, then understanding the *meaning* of these ontic quantitative properties requires *human intervention*; it needs the reflective and constructive disclosure and opening-up of this meaning through the formation of mathematical theories. If so, then what is the relationship between these ontic quantitative properties and the particulars of our (numerically specified) answer to the question: how many? To put it differently: in order to relate to a given unity or multiplicity humans have to grasp this quantitative meaning with the aid of numerals (*numerical symbols*). This entails that we can *identify* and *designate* ontic quantitative properties in assigning numbers to them through this act of identification and designation.

With an appeal to Gonseth we find an account of the subject matter of mathematics in the thinking of Bernays which does want to maintain a connection between experianceable things ("phänomenalen Gegenständlichkeit") and the contents acquired

16 "Gegen dieses axiomatische Vorgehen besteht auch nicht etwa der Vorwurf der Willkürlichkeit zu Recht, denn wir haben es bei den Grundlagen der systematische Arithmetik nicht mit einem beliebigen, nach Bedarf zusammengestellten Axiomensystem zu tun, sondern mit einer *naturgemäßen systematischen Extrapolation der Elementare Zahlenlehre*" (Bernays, 1976:45).

17 "Es scheint, daß nur eine vorgefasste philosophische Ansicht dieses Erfordernis bestimmt, die Ansicht nämlich, daß es nur *eine* Art von Tatsächlichkeit geben könne, diejenige der konktreten Wirklichkeit" (Bernays, 1976:122).

through processes of idealization and abstraction (this content has a *structural* character – Bernays, 1976:180).[18] In particular mathematical idealization comes to expression through axiomatization (Bernays, 1976:181).

The fact that Bernays, as we have mentioned above, questions the claim that there is only *one kind of factuality*, namely that of the "concrete," urges us to investigate this issue in more detail, with particular reference to the conceptions of Gödel, Wang and Cassirer.

Gödel introduces the idea of "semiperceptions" when it concerns "mathematical objects." Next to a physical causal context within which something can be "given," Gödel refers to data of a second kind which are open to "semiperceptions." The data of this second kind "cannot be associated with actions of certain things upon our sense organs" (quoted by Wang, 1988:304). Gödel says:

> It by no means follows, however, [that they] are something purely subjective as Kant says. Rather they, too, may represent '*an aspect of objective reality*' (my emphasis – DS), but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality (quoted by Wang, 1988:304).

Wang is "inclined to agree with Gödel," but he does "not know how to elaborate his assertions" (Wang, 1988:304). He says that he "used to be troubled by the association of objective existence with having a fixed 'residence' in spacetime," but he now feels "that 'an aspect of objective reality' can exist (and be 'perceived by semiperceptions') without its occupying a location in spacetime in the way physical objects do" (Wang, 1988:304).

Clearly, Gödel and Wang contemplate the "reality" of "ontic" (designated by them as: "objective") "aspects of reality" which are not like "concrete entities" occupying "a location in spacetime."

If we connect this kind of *non-entitary reality* with the (above-mentioned) arithmetical question formulated by Frege, namely the fundamental *numerical* question *How many?*, [19] then some significant insights of Cassirer may be helpful in this context.

His basic concern in 1910 is to make a distinction between *entity* (designated by him as: *substance*) and *function*. Although used in a slightly different context, his employment of the terms "what" and "how" (1953:40) may be elucidating. An answer to the *what*-question results in the identification of an entity (or a certain *kind* of entity), whereas the answer to the *how*-question specifies a function (aspect) of reality. Cassirer approximates the position of Frege closely even though he uses the notion of "abstraction." He does that, however, while having in mind not merely the entitary-directed kind of abstraction against which Frege argued. The act of abstraction has as its aim "to bring out the meaning of a certain relation independently of all particular cases of application" (Cassirer, 1953:39). Keeping in mind that "relational concepts" are for Cassirer the same as *function concepts*, his following statement is significant for the distinction between aspects (functions) and entities (things, so-called "objects"):

18 "Die mathematische Gegenständlichkeit geht durch Idealisierungs- und Abstraktionsprozesse hervor aus der phänomenalen Gegenstandlichkeit des *Strukturellen*."

19 As an answer to the question "How many objects of a certain kind are there" (Frege, 1903:§157 and Dummett, 1995:64).

> The fucntion of 'number' is, in its meaning, independent of the factual diversity of the objects which are enumerated. ... Here abstraction ... means logical concentration on the relational connection as such (Cassirer, 1953:39).[20]

Surely, the mentioned considerations of Gödel, Wang and Cassirer call for an acknowledgment of *a function* or *an aspect* of reality which is *ontically given* but not given *in the same way* as *physical* (or other kinds of) *entities*. Understanding the meaning of the *ontically given aspects* or *functions* indeed requires a formative (constructive) human activity, in its scholarly sense performed by mathematicians who are capable, through their articulation of *mathematical theories*, to make this given numerical meaning explicit and to disclose it in their erection of mathematical structures. This perspective is an argument in opposition to both *platonism* and *pure constructivism*. Against platonism it acknowledges that thinking about the meaning of number does not simply rest on the acceptance of an already existing transcendent (ideal) world of "mathematical objects" of the thinking human mind, since without the intervention of human thinking the ontically given meaning of quantity (plurality) cannot be *disclosed* and *articulated* in *mathematical structures*. By contrast, it also opposes *constructivism* in acknowledging that the *subject matter* of mathematics is not merely the product of the thought-activities of mathematicians, because such activities presuppose at least "an aspect of objective reality."

So-called "evolutionary epistemology" runs into serious difficulties if it wants to explain the ontic givenness of *plurality* (multiplicity), because long before the (paleontological) appearance of human beings, entities (note the *plural* form: *entities*!) did function in the quantitative mode of reality, did occupy space, did move, and so on. Furthermore, if this epistemology starts from *sense-perception* the only option open is that of *entitary-directed abstraction* – which is criticized by Frege as being incapable of justifying our concept of number. We have seen that another kind of abstration is required, namely *modal abstraction*. The property argued for in the next paragraph, namely (modal) *universality*, is also inexplicable merely in terms of sense-perception.

The universality of functional modes

We have noted that when Frege employs the term 'logical' (as in the phrase: "logical objects"), he always has the following distinctive feature of "the logical" in mind – in the words of Dummett: "its generality: it does not relate to any special domain of knowledge, for, just as objects of any kind can be numbered, so objects of any kind can belong to a class" (Dummett, 1995:224).

This view of Frege actually pertains to a common feature of each and every *aspect* of reality, not only to the *logical aspect*. It is part of the nature of an aspect or *modus* that its scope transcends the multiplicity of individual things merely *functioning* within it. This constitutes what ought to be designated as the *modal universality* to which we have alluded above – amongst others also holding for the *quantitative aspect of reality*. This modal universality of the numerical aspect clearly surfaces in Frege's account, as explained by Dummett:

> A correct definition of the *natural numbers* must, on his view, show how such a number can be used to say how many matches there are in a box or books on a shelf. Yet number theory has nothing to do with matches or with books: its

20 Here Cassirer highlights what we have called *property-abstraction* (*modal abstraction*).

> business in this regard is only to display what, in general, is involved in stating the cardinality of objects, of whatever sort, that fall under some concept, and how natural numbers can be used for this purpose (Dummett, 1955:272).

At this point Frege (and Dummett) were on the verge of the alternative theory which we have in mind when we speak about *modal universality* – if it was not the case that Frege identified the ontic meaning of the arithmetical aspect of reality with the nature of a concept. This theory first of all eludicates the *uniqueness* of functions as distinct from *entities*. In conceptual thought this difference surfaces in the distinction between the *concept of number* and *type concepts*.

Whenever (empirical) investigation, with the aid of *entitary-directed abstraction* as explained, arrives at the concept of this or that *type* of entity, there is an inherent restriction to the *number* of entities involved, namely only to those *belonging to that type*.

Stated differently, i.e., in terms of a *law-perspective*: the *type-law* of a specific kind of entities only holds for that *type of entity*, i.e., for a *limited class* of entities. For example, the *law for being an atom* is limited to atoms and does not apply to molecules or planets. Similarly, the *Coulomb law* (applicable only to charged physical entities) and the *Pauli principle* (applicable to fermions) do not hold for all possible (physical) entities. Rather than delineating a *limited* class of entities, "property laws" (i.e., *modal functional laws*), however, have a generality (*universality*) which embraces all possible entities because they "describe a mode of being, relatedness, experience, or explanation" (as phrased by Stafleu, 1980:11). The fact that modal laws – such as those of quantum physics – hold for all possible "objects," is clearly observed by the German physicist Von Weizsäcker: "Quantum theory, formulated sufficiently abstract, is a universal theory for all *Gegenstandklassen* (classes of objects)" (1993:128). When he explains, on the next page, that one cannot deduce the *kinds* of entities of experience from the *universal scope* of quantum theory, he gives his own account of what we are designating as *type laws*.

This distinction shows a similarity with the way in which Frege employs the word *quantity*. Dummett writes:

> Frege so uses it that a phrase like '2.6 metres' designates a specific quantity of one kind, '5.3 seconds' a quantity of another kind, and so on. He thus takes quantities to be objects, distinct from numbers of any kind. There cannot be two equal quantities, on this use: if two bodies are equal in mass, they have the *same* mass. Quantities fall into many distinct types: masses form one type, lengths another, temperatures a third (Dummett, 1995:270).

Frege implicitly distinguishes between the general (modally universal) meaning of *number* and the *specifications* it receives when it is attached to different *types of quantities* – in which case he does not speak about *number* but about *quantity*.

Compare in connection with our notion of modal universality also the position of Russell and Gödel. From Russell's *Introduction to Mathematical Philosophy* Gödel quotes the second half of the following sentence: "Logic, I should maintain, must no more admit a unicorn than zoology; for logic is concerned with the *real world* (my emphasis – DS) just as truly as zoology, though with its more abstract and general features" (Wang, 1988:313). Whereas zoology has a foremost interest in living entities (animals), logic, with its concern for the "more abstract and general," operates on the level of *modal universality*.

In yet another fashion Dummett implicitly alludes to the nature of modal universality in the following explanatory statement:

> The contrast between arithmetical and empirical enquiry concerns not so much the discovery of individual objects as the delineation of the area of search. The astronomer need have no precise conception of the totality of celestial objects: he is concerned with detecting whatever is describable in physical terms and lies, or originates, outside the earth's atmosphere, and he need give no further specification of this 'whatever'. In mathematics, by contrast, an existential conjecture, to have any definite content, requires a prior circumscription of the domain of quantification (Dummett, 1995:228).

Compare another instance of some remarks (the second one already mentioned in a similar context) by Dummett intended to explain Frege's concept of number, which also explicitly highlights this feature of *modal universality* (the universal scope of the numerical aspect of reality, in the thinking of Frege – mediated by the nature of "the logical / a concept"):

> In view of the generality of number – the fact that there is no restriction on the type of objects of which we can say how many there are – ... (Dummett, 1995: 73); The term "logical", in the phrase "logical objects", refers to what Frege always picked out as the distinguishing mark of the logical, its generality: it does not relate to any specific domain of knowledge, for, just as objects of any kind can be numbered, so objects of any kind can belong to a class (Dummet, 1995:224).

Given his logicist prejudice Frege did not realize that his notion of the *general applicability* of arithmetic actually points in the direction of the *modal* or *functional universality* of the *arithmetical* aspect of ***reality***. This means that the logical and the arithmetical modes both share this feature of *modal universality*. However, as we shall see, the *foundational* position of the numerical aspect with respect to the logical aspect justifies the view that the mathematical treatment of the arithmetical represents a higher level of abstraction. Bernays also defends this position: "However, in respect of the formal, the mathematical perspective, as opposed to the logical conceptual one, represents the standpoint of a higher abstraction".[21]

The implication of the stance taken by Bernays is that the arithmetical mode is foundational to the logical mode and that it is therefore not possible to reduce arithmetic to logic.

The cul de sac of logicism

Bernays noticed that whereas both Frege and Dedekind throughout their mathematical reflections and proofs evinced the most sincere and strict precision, they were wholly negligent in assuming the representation of a closed totality of all possible conceivable logical objects.[22]

21 "In Hinsicht auf das Formale stellt aber, ..., die mathematische Betrachtung gegenüber der begrifflich logischen den Standpunkt der höheren Abstraktion dar" – Bernays, 1976:27.

22 "So waren Frege und Dedekind, deren Beweisführungen und Überlegungen sonst überall durch äußerste Präzision und Strenge ausgezeichnet sind, ganz unbedenklich in dem, was sie als vermeintlich selbst-

David Hilbert also points at the dilemma entailed in the logicist attempt to deduce the meaning of number from that of the logical-analytical mode. In his *Gesammelte Abhandlungen* Hilbert writes:

> Only when we analyze attentively do we realize that in presenting the laws of logic we already had to employ certain arithmetical basic concepts, for example the concept of a set and partially also the concept of number, particularly as cardinal number [Anzahl]. Here we end up in a vicious circle and in order to avoid paradoxes it is necessary to come to a partially simultaneous development of the laws of logic and arithmetic (1970:199).

Cassirer first of all approaches this problem in terms of the numerical analogy within the logical-analytical aspect. His question is that it is not understandable why one only accepts *logical* identity and diversity, which enter the set concept as necessary elements, as such basic functions, but that one does not do the same with regard to *numerical* unity and difference. He claims that a truly satisfactory deduction of the one from the other is also not achieved by set theory, which entails a persistent suspicion that all similar attempts will continue to harbor a concealed epistemological circle.[23] That the logicist position taken by Frege is unsound was brought into the open when Russell made the absurdity of set theory known in 1900. Consider the set C with elements A and the prescription that elements of set C may only be *those* sets A which do not contain themselves as elements.

Thus C = (A/A   A). (The set of ten chairs is e.g. not itself a chair and does not contain itself as an element. On the other hand the set of thinkable thoughts *is* in itself thinkable and therefore does contain itself as an element.) Now suppose that C is an element of C (C   C). Every element of C, however, does not contain itself as an element – this, after all, is the requirement for being an element of C. This implies that if C is an element of C, it must *also* meet this requirement – but then C   C implies C   C! Suppose on the other hand that C   C. Then C *does* meet the requirement for being an element of C, which means that C   C. In other words, C is an element of C if and only if C is not an element of C.[24]

Dummett mentions that Frege discovered by August 1906 that the flaw in his account cannot be corrected within the framework of his theory.[25]

The assessment given by Dummett is straight-forward:

verständliche Voraussetzung dem Standpunkt der allgemeinen Logik zugrunde legten, nämlich in der Vorstellung von einer abgeschlossenen Gesamtheid aller überhaupt denkbaren logischen Objekte" (Bernays, 1976:47).

23 "In der Tat ist nicht einzusehen, warum man lediglich logische Identität und Verschiedenheit, die als notwendige Momente in den Mengenbegriff eingehen, als solche Urfunktionen gelten lassen und nicht auch die numerische Einheit und den numersichen Unterschied von Anfang an in diesen Kreis aufnehmen will. Eine wirklich befriedigende Herleitung des einen aus dem anderen ist auch der mengentheoretischen Auffassung nicht gelungen, und der Verdacht eines versteckten erkenntnistheoretischen Zirkels blieb gegenüber allen Versuchen, die in dieser Richtung gemacht werden, immer bestehen" (Cassirer, 1957:73-74).

24 Of course this conclusion presupposes the application of the principle of the *Excluded Middle* (i.e., the acceptance of *infinite totalities*) – something rejected by intuitionistic mathematics.

25 That is, "with the abstraction operator as primitive and an axiom governing the condition for identity of value-ranges: but the underlying error lay much deeper than a misconception concerning the foundations of set theory. It was an error affecting his entire philosophy" (Dummett, 1995:223).

> Frege had answers – by no means always the right answers – to all the philosophical problems concerning the branches of mathematics with which he dealt. He had an account to offer of the applications of arithmetic; of the status of its objects; of the kind of necessity attaching to arithmetical truths; and of how to reconcile their a priori character with our attainment of new knowledge about arithmetic. His view of the status of the numbers, ontological and epistemological, proved to be catastrophically wrong; for the last nineteen years of his life, he himself acknowledged it to have been wrong, and regarded that as bringing with it the collapse of his entire philosophy of arithmetic. In spite of efforts like those of Crispin Wright to defend it, we can clearly see that his view of this question was in error: but we have not supplied any very good alternative. In answering the remaining questions, we have not, save in one crucial respect, advanced very far beyond Frege at all (Dummett, 1955:292).

Logicism had to concede that the existence of an *infinite totality* cannot be proven. Consequently, infinity was introduced by an *Axiom of Infinity*.[26]

This concession had another consequence: no longer accepting the idea of the totality of all logical "objects" entails that also the acceptance of the totality of predicates became problematic.[27]

Once again: The "miracle" of applicability

In conclusion we have to return briefly to the issue of the applicability of mathematics.

In his work on "Warrant and Proper Function" Plantinga (1993:232, note 2) elucidates this "miracle" or "mystery" as follows:

> This hasn't been lost on those who have thought about the matter. According to Erwin Schrödinger, the fact that we human beings can discover the laws of nature is "a miracle that may well be beyond human understanding" (What is Life? [Cambridge: University of Cambridge Press, 1945], p.31). According to Eugene Wigner, "The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious, and there is no rational explanation for it" ("The Unreasonable Effectiveness of Mathematics in the Natural Sciences," in *On Pure and Applied Mathematics*, [13, p.2]) and "It is difficult to avoid the impression that a miracle confronts us here, quite comparable in its striking nature to the miracle that the human mind can string a thousand arguments together without getting itself into contradictions, or to the two miracles of the existence of laws of nature and of the human mind's capacity to derive them" (p.7). And Albert Einstein thought the intelligibility of the world a "miracle or an eternal mystery" (Lettres à Maurice Solouine [Paris: Gauthier-Villars, 1956], p.115).

26 Cf. Bernays: "Die Logistik verzichtet seither darauf, die Existenz einer unendlichen Gesamtheit zu beweisen und stellt vielmehr ausdrücklich ein *Unendlichkeitsaxiom* auf" (Bernays, 1976:47). Fraenkel *et al* remark: "It seems, then, that the only really serious drawback in the Frege-Russell thesis is the doubtful status of InfAx [InfAx = Axiom of Infinity – DS], according to the interpretation intended by them" (1973:186).

27 Cf. the remark of Bernays: "Mit der Preisgabe der Vorstellung von der Gesamtheit aller logischen Gegenstände wird aber auch die Vorstellung von der Gesamtheit aller Prädikate problematisch, und bei näherem Zusehen zeigt sich hierin eine besondere grundsätzliche Schwierigkeit" (Bernays, 1976:47).

We have quoted Dummett's account of Frege's position: "A correct definition of the *natural numbers* must, on his view, show how such a number can be used to say how many matches there are in a box or books on a shelf. Yet number theory has nothing to do with matches or with books: its business in this regard is only to display what, in general, is involved in stating the cardinality of objects, of whatever sort, that fall under some concept, and how natural numbers can be used for this purpose" (Dummett, 1955:272).

As soon as one acknowledges that "objective reality" displays an *ontically given* numerical (arithmetical) **aspect**, the quoted phrase: "Yet number theory has nothing to do with matches or with books" has to be corrected. Material things – and whatever else belong to the dimension of entities within reality – invariably *function* within the numerical (and other) modes/aspects of reality. In so far as "matches" or "books on a shelf" therefore function within the numerical mode, this *functioning* has everything to do with the meaning of this aspect!

But if entities cannot avoid to have specific functions within the various modes of reality, it stands to reason that the universal modal laws obtaining within the numerical (and spatial) aspect(s) will hold for every kind of entity functioning within this (these) mode(s) – notwithstanding the fact that, for example, the application of arithmetical theories may acquire peculiar *specifications* within the context of different types of non-arithmetical relations or entities.[28] This affirmation, however, presupposes the *distinctness* and *mutual coherence* between the dimensions of modal aspects and that of concretely existing entities.

Because "(modal) abstraction" ultimately does not cut all ties with "reality," mathematics will remain *applicable* – albeit for reasons different than those sustained by Frege and other platonists.[29]

In addition it should be pointed out that the applicability of, for example, arithmetical insights, is not only dependent upon the fact that the arithmetical mode conditions whatever functions within it in a quantitative way, since other aspects of reality also stand in a relation of *mutual coherence* with the numerical aspect. This already surfaced when we quoted Hilbert in respect of the fact that the development of logic is founded in a simultaneous development of arithmetic. To bring it closer to the "home" of arithmetic: topology and geometry, in their analysis of the meaning of space, are inconceivable without the foundation of number, as is seen from terms such as (spatial) *magnitudes* and (spatial) *dimensions* – which undeniably reflect the foundational meaning of number. Although topology disregards metrical properties, it still employs the notion of *open* sets, which alludes to the spatial mode.

The interconnection between the arithmetical and the logical mode, furthermore, comes to expression in the logical subject-object relation. If the core meaning of the analytical mode (the logical aspect) is given in *identification* and *distinguishing*, then every act of identification objectifies within the logical mode whatever is identified and distinguished. *Objectification* is an act of an analytical subject. Surely, this activ-

28 Just recall Frege's distinciton between *number* and *quantities* (cf. Dummet 1995:270).

29 Keeping in mind the "non-logicist platonism" of Gödel (cf. Dummett, 1955:301), we may here refer to Dummett's words: "Logicism is not the most natural ally of platonism, because, on the most natural view of logic, there are no logical objects: it was a tour de force on Frege's part to combine a vehement advocacy of platonism with an unreserved logicism about number theory and analysis" (Dummett, 1955: 301).

ity is *normed*, which means that one can arrive at logically correct or incorrect (norm-conforming or antinormative) acts of objectification. *Identification* is actually nothing but the way in which we form a *concept* of something. A concept is constituted as a *logical unity* in the *multiplicity* of – (universal) features *unified* in it. In order to form a concept of number, logical acts of identification and distinction are involved.

What then about Frege's contention that there are not *distinct* numbers one, but only a *single one*, which is incapable of a *plural*?

Frege here emphasizes *logical identity* at the cost of the implied *universal side* of the property involved. Wherever the number "one" is applied to something, or wherever theoretical reflection enters into an analysis of the meaning of the number "one," the *universal orderliness* of this number comes to expression. The *conceptual* "oneness" of the number "one" relates to its *universality*, its *orderliness*, i.e., to the way in which every instantiation of "oneness" testifies to the underlying condition for "being-one," which makes it possible in the first place.[30]

> The primary issue is therefore that of *universality* and *specific instances* and not that of *one* and *many* (*plurality*) as such.

Owing to his logicist bias, Frege inverted the order of foundation of the logical mode. Number does not in the first place attach to a concept – the primary vehicle of *logical objectification* – because the quantitative meaning of number can only be *objectified* in acts of *identification*, i.e., through the formation of *number concepts*. But the original domain to which the meaning of number "attaches" is the quantitative aspect of "objective reality" (in its ontic "givenness"). Only in a secondary (derived, analogical) sense does it relate to *concepts.*

In his attempt to develop number theory by eliminating mathematical primitives, Zalta mentions Frege's question: "How do we apprehend numbers given that we have no intuitions of them?" (1999:620). His theory concerns numbers which "are abstracted from the facts about concrete objects" and he adds that "the resulting numbers are even more closely tied to their application in counting the objects of the natural world than Frege anticipated" (1999:620). However, he has promised to discuss these philosophical issues more fully at another occasion. What is nonetheless clear from his current article is that he does not realize that the underlying logic employed by him *presupposes* the *primitive meaning* of all the pre-logical aspects of reality. (See the concluding remark of this article.)

Every logical unity and multiplicity presuppose a (foundational) arithmetical unity and multiplicity. In fact, within every aspect of reality *distinct* from the numerical one there are *analogical specifications* of the original quantatitive meaning of number. For example, within space we meet vector addition (where 3 + 4 equals 5 instead of 7); within the kinematical aspect Einstein employed the Lorentz transformations in terms of which 0.9c + 0.9c equals 1.80/1.81c (in stead of 1.8c); and so on.

The numerical aspect is therefore related to entitary reality through its *modal universality* – evidenced in the fact that every entity whatsoever has a function within the arithmetical aspect – explaining the *applicability* of number theory, while it is connected to all other aspects of reality through its *analogical appearances* within each of them.

30 Similarly, wherever we encounter "gold" it is an instantiation of the the (universal) conditions for *being-gold*.

Concluding remark: A theory of modal aspects

Having reached this stage of our argument, the natural thing would be to explore more fully the basic elements of a constructive theory of modal aspects (functions). Up to this point the issues addressed were discussed in terms of the emerging problem of *non-entitary* ontic properties. In order to provide a first assessment it was necessary to introduce a minimal explanation of a theory of modal aspects.

However, since explaining such a theory in more detail will exceed the limits of the present article, the best option seems to be to treat such a theory in a follow-up article. As a point of departure this article will commence by investigating a well-known expression within the English language: "Physical Objects?" The question mark suggests that the theory of modal aspects entails the argument that insofar as material things are *physical* they are not objects but *subjects*, and insofar as they are *objects* they are not physical for in the latter case they ought to be understood according to *non-physical* properties (so-called *object-functions*). Elucidating the grounds for this claim will then open the way to return to the outcome of our analysis of the position of Frege in this article. What will then be explained in more detail are the criteria underlying the assertion of non-entitary *ontic* aspects of reality.

Literature

Behnke, H., Bertram, G., Collatz, L., Sauer, R., & Unger, H. (eds.) (1968): *Grundzüge der Mathematik*, Volume V, Praktische Methoden und Anwendungen der Mathematik, Göttingen : (Publisher not known).

Cantor, G. (1962): *Gesammelte Abhandlungen Mathematischen und Philosophischen Inhalts*, Hildesheim (1932).

Cassirer, E. (1910): *Substanzbegriff und Funktionsbegriff*, Berlin (1910), Darmstadt 1969.

Cassirer, E. (1953): *Substance and Function*, New York 1953 (first edition of the English translation of *Substanzbegriff und Funktionsbegriff*: 1923. First German edition 1910).

Cassirer, E. (1957): *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit*, Stuttgart: Kohlhammer Verlag.

Cassirer, E. (1958):The Library of Living Philosophers, *The Philosophy of Ernst Cassirer*, edited by P.A. Schilpp, New York: Tudor Publishing Company, First edition (second printing).

Dedekind, R. (1901): *Essays on the Theory of Numbers*, Chicago.

Dedekind, R. (1887): *Was sind und was sollen die Zahlen*, Braunschweig (10th ed.) 1969.

Dummett, M.A.E. (1995): *FREGE, Philosophy of Mathematics*, Harvard University Press, Cambridge, Second Printing.

Felgner, U. (Editor) (1979): *Mengenlehre*, Wissenschaftliche Buchgesellschaft, Darmstadt.

Fraenkel, A., Bar-Hillel, Y., Levy, A. & Van Dalen, D. (1973): *Foundations of Set Theory*, 2nd revised edition, Amsterdam; North Holland.

Frege, G. (1884): *Grundlagen der Arithmetik*, Breslau: Verlag M & H. Marcus ("Unveränderter Neudruck" – unaltered reprint, 1934).

Frege, G. (1893): *Grundgesetze der Arithmetik*, Vol.I. Jena.

Frege, G. (1903): *Grundgesetze der Arithmetik*, Vol.II Jena.

Freudenthal, H. (1940): Zur Geschichte der vollständigen Induktion, *Archives Internationales d'Histoire des Science*, Vol. 22.

Hilbert, D. (1925): Über das Unendliche, *Mathematische Annalen*, Vol. 95, 1925 (pp.161-190).

Hilbert, D. (1930): Naturerkennen und Logik, in: Hilbert, 193 (pp.378-387).

Hilbert, D. (1935), *Gesammelte Abhandlungen*, Vol.3, Second Edition, Berlin: Springer 1970.

Husserl, E. (1970): *Philosophie der Arithmetik* (1891), Husserliana, The Hague: Martinus Nijhof, Vol. XII.

Kattsoff, L.O. (1973): On the Nature of Mathematical Entities, in: *International Logic Review*, Number 7, 1973 (pp.29-45).

Meschkowski, H. (Editor) (1972): *Grundlagen der modernen Mathematik*, Darmstadt.

Plantinga, A. (1993): *Warrant and Proper Function*, Oxford: Oxford University Press.

Russell, B. (1956): *The Principles of Mathematics*, London (1903).

Schilpp, P.A. (Editor) (1958): *The Library of Living Philosophers, The Philosophy of Ernst Cassirer*, New York: Tudor Publishing Company, First edition (second printing).

Simpson, G.G. (1969): *Biology and Man*, New York.

Skolem, Th. (1922): Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre, in: Felgner, pp.57-72, 1979.

Skolem, Th. (1929): *Über die Grundlagendiskussionen in der Mathematik*, in: Felgner, pp.73-91, 1979.

Von Weizsäcker, C.F. (1993): *Der Mensch in seiner Geschichte*, München : DTV.

Weyl, H. (1946): Mathematics and Logic, A brief survey serving as preface to a review of The Philosophy of Bertrand Russell, *American Mathematical Monthly*, Vol. 53.

Wang, Hao (1957): The Axiomatization of Arithmetic, in: *Journal of Symbolic Logic*, Volume 22, pp.145-157.

Wang, Hao (1988): *Reflections on Gödel*, MIT Press, Cambridge, Massachusetts.

Weyl, H. (1966): *Philosophie der Mathematik und Naturwissenschaft*, 3rd revised and extended edition, Vienna.

Zalta, E.N. (1999): Natural numbers and natural cardinals as abstract objects: a partial reconstruction of Frege's *Grundgesetze* in object theory, in: *Journal of philosophical logic*, Volume / Issue 28, 6, pp.619-660.