Bernays, Dooyeweerd and Gödel – the remarkable convergence in their reflections on the foundations of mathematics

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Abstract:
In spite of differences the thought of Bernays, Dooyeweerd and Gödel evinces a remarkable convergence. This is particularly the case in respect of the acknowledgement of the difference between the *discrete* and the *continuous*, the foundational position of number and the fact that the idea of continuity is derived from space (geometry – Bernays). What is furthermore similar is the recognition of what is *primitive* (and *indefinable*) as well as the account of the *coherence* of what is *unique*, such as when Gödel observes something quasi-spatial character of sets. It is shown that Dooyeweerd's theory of modal aspects provides a philosophical framework that exceeds his own restrictive understanding of infinity)to the potential infinite) and at the same time makes it possible to account for key insights found in the thought of Bernays and Gödel. When Laugwitz says that discreteness rules within the sphere of the numerical, he says nothing more than what Dooyeweerd had in mind with his idea that *discrete quantity*, as the meaning-nucleus of the arithmetical aspect, *qualifies* every element within the structure of the quantitative aspect. And when Bernays says that analysis expresses the idea of the continuum in arithmetical language his mode of speech is equivalent to saying that mathematical analysis could seen as being founded upon the spatial anticipation within the modal structure of the arithmetical aspect. The view of the actual infinite (the at once infinite) in terms of an "as if" approach (Bernays), that is, as appreciated as a regulative hypothesis through which every successively infinite multiplicity of numbers could be envisaged as being giving all at once, as an infinite totality, provides a sound understanding of the *at once infinite* and makes it plain why every form of arithmeticism fails. Such attempts have to call upon Cantor's proof on the non-denumerability of the real numbers – and this proof pre-supposes the use of the at once infinite which, in turn, pre-supposes the (irreducibile) spatial order of simultaneity and the patial whole-parts relation.

In order briefly to contextualize our reflections, a few preliminary remarks are required before we investigate the remarkable convergence specified in its title.

The issues discussed in this article primarily focus on the mathematical philosophy of Bernays, accounting for twelve key distinctions found in his thought (listed in § 3 below). One of these twelve issues (namely number 5 regarding what is ontically given) borders upon two articles published earlier in the *South African Journal of Philosophy* (SAJP) (see Strauss, 2003 and Strauss 2006) – although the current investigation takes a different course by providing an extensive analysis of the thought of Bernays which is *completely new*. This analysis then serves as the basis for comparing the ideas of Dooyeweerd and Gödel with those deveoped by Bernays. No background knowledge is presupposed regarding the philosophical views of anyone of these three scholars. Every distinction advanced and every insight discussed is explained in context whenever deemed necessary for an understanding of the argument. The positive appreciation of the thought of Bernays opened up the possibility to criticize a fundamental shortcoming in Dooyeweerd's approach, found in his rejection of the actual infinite (influenced by die intuitionism of Brouwer and Weyl). In addition the approach advaneced by Bernays fundamentally questions the prevailing arithmeticism in the dominant schools of thought of modern mathematics. Bernays disqualified this arithmeticism as an "arbitrary thesis" and opted for an "as if" approach regarding the use van die actual inifnite in mathematical analysis. In fact he makes an appeal to irreducible features of space.

Although it is not the aim of this article to discuss *all* the current issues that are still alive in literature on the foundations of mathematics, all the issues discussed in this article are still found within contemporary publications on the foundation of mathematics. For example, when Fraenkel *et al*. hold that "[B]ridging the gap between the domains of discreteness and of continuity, … is a central, presumably even the central problem of the foundation of mathematics" (Fraenkel, 1973:211) it is mentioned in footnote 10 that more recently Maddy (see Maddy, 1997:14, 39-42, 47-48, 50, 52-54, 57-58, 61, 85) still considers this work of Fraenkel *et al*. as *authoritative*. The switch from arithmeticism to geometricism in Greek mathematics is repeated by Frege before the end of his life (see Frege, 1979:277), and in 2001 it has been followed up by a group of French mathematicians, amongst them Longo and Thom mentioned in the text and footnote 12 (they hold the view that the *continuum* ontologically precedes the discrete, degrading the latter to be a *catastrophe*, a *broken* line). Recently Shapiro edited *The Oxford Handbook of Philosophy of Mathematics and Logic* (2005) – and every issue found in the thought of Bernays (co-worker of the foremost mathematician of the 20th century, David Hilbert) is still present within the discussions found in this 833 page work. Tait's work on *The Provenance of Pure Reason, Essays in the Philosophy of Mathematics and Its History* (2005) also engages in the issues discussed in this article, showing that apart from the significance of comparing the views of Bernays, Dooyeweerd and Gödel, the problems surfacing in this comparison also relate to current foundational discussions. Tait illuminates shortcomings in Frege's understanding of number that practically coincides with my own position (see Tait, 2005:241 and Strauss, 2009:51).

Sometimes the views of contemporary authors who work on the foundations of mathematics are quoted in connection with the key points argued for in this article. In addition to Obojska mentioned in the next paragraph, the name of Laugwitz comes to mind. He emphasizes that even in Cantor's circumscription of a set the "discrete rules" (see Laugwitz, 1986 and 1997).

Recent developments in subdiscipline of mathematics known as *mereology* highlight the inevitability of *primitive* numerical and spatial terms, for example when Obojska intends to introduce a *single primitive notion* as basis for mereology, namely (*primary*) *relation* (see Obojska, 2007:644, 646). The term "primary" is derived from the core meaning of number ("first") while the term "relation" is derived from the core meaning of space, for to be related is to be connected and to be connected entails coherence which is synonymous with continuity which in turn is synonymous with the original spatial whole-parts relation. She says that Cantor's theory is distributive (starting with points as elements considered as a whole) while Lesniewski's approach is collective because "a mereological 'set' is a whole (a collective aggregate or class) composed of 'parts' and the fundamental relation is that of being a 'part' of the whole, an element of a class" (Obojska, 2007:642). In respect of the feature of being coherent Shapiro mentions that unlike isomorphism "coherence is not a rigorously defined mathematical concept, and there is no noncircular way to characterize it" (Shapiro, 1997:13).

1. Orientation

The three thinkers mentioned in the title of this article were contemporaries, two of them died in the same year, 1977, and the other one a year later in 1978.[1]

Bernays was a full-blown mathematician who studied under formidable figures such as David Hilbert, Edmund Landau, Hermann Weyl and Felix Klein (teaching at Berlin and Göttingen). Noteworthy is his philosophical studies under Alois Riehl, Carl Stumpf and in particular Ernst Cassirer who, as we shall see below, also had a crucial influence upon the thought of Dooyeweerd. He received his Ph.D from Berlin and then completed his "Habilitationsschrift" at the University of Zürich. Ernst Zermelo, who introduced the first group of axioms for set theory, was the examiner. After he completed his Ph.D in 1912 he worked as an assistant of Zermelo at the university of Zürich until 1917. Later on Bernays also developed his own system of axioms for set theory. Since 1917 Bernays started to work with Hilbert (1862-1943) on the foundations of mathematics. This relationship eventually produced a two volume work which they jointly published (in 1934 and 1939): "Grundlagen der Mathematik."

Gödel, who grew up in Austria, is rated in the first place as an outstanding logician, although his discoveries had a direct impact on mathematics, in particular on consistency and completeness.[2] In 1930, when Hilbert received honorary citizenship from his birth place, Königsberg, he concluded his oration on *knowledge of nature and logic* ("Naturerkennen und Logik") with the optimistic lines: "Wir müssen wissen, Wir werden wissen" ("We must know, we shall know" – Hilbert, 1970:387 – it now serves as the inscripturation on his tombstone). This hope included the expectation that it would be possible to demonstrate that the axiomatic foundation of mathematics se-

1 Bernays – 1888-1977; Dooyeweerd – 1894-1977 and Gödel 1906-1978. Bernays reached an age of 88, Dooyeweerd 82 and Gödel 75.

2 On request from the Journal *Erkenntnis* he produced a summary formulation of his famous 1931 article on "Über formal unentscheidbare Sätze der *Principia mathematica* und verwandter Systeme." His succinct statement of the problems treated in this article reads: "The paper deals with problems of two kinds, namely 1. the question of completeness (decidability) of formal systems of mathematics, 2. the question of consistency proofs for such systems" (Gödel, 1986:202). ["Es handelt sich in dieser Arbeit um Probleme von Zweierlei Art, nämlich 1. Um die Frage der Vollständigkeit (Entscheidungsdefinitheit) formaler Systeme der Mathematik, 2. Um die Frage der Widerspruchsfreiheitsbeweise für solche Systeme".]

cures the consistency of these axioms excluding any contradictions as well as their completeness enabling the demonstration of all true statements. Yet Gödel shattered both these expectations. Any consistency proof will exceed the axioms of the system and any axiomatic system capable of axiomatizing arithmetic contains unprovable but true statements. Although he appreciated employing the idea of *infinite totalities*, Gödel with equal clarity stressed the fact that constructing sets through the iterative application of the operation "set of" has "never led to any antinomies." Intuitive, unformalized arithmetic, during the vast history of mathematical thinking, employed this basic notion of succession in an unproblematic way, without "dividing the totality of all existing things into two categories" (quoted by Yourgrau, 2005:137).

This (as we shall see below – *restricted*) understanding of the infinite is also found in the thought of Herman Dooyeweerd. In 1917 he completed his legal studies on a dissertation from the field of law, entitled: *De Ministerraad in het Nederlandsche staatsrecht* ("The Cabinet in Dutch constitutional law"). During the early 1920s he broadened his perspective by contemplating general philosophical problems and their significance for the various academic disciplines, in particular the science of law, and including an in-depth study of the history of philosophy. In developing a novel ontology the most significant contribution it made towards an analysis of the foundations of the special sciences is found in his *theory of modal aspects*. This theory provides a new systematic way to account for *primitive terms*, for the problem of *uniqueness* and *coherence*, as well as for the *interconnections* (moments of coherence or modal analogies) between the domains harbouring what are primitive and unique (we shall see that Gödel also sensed the importance of these issues). By attaching his understanding of infinity to the intuitionistic restriction of mathematics, that is, to the *potential infinite*, Dooyeweerd unfortunately did not explore the enriching implications of his own theory of modal aspects in connection with the infinite (we shall return to this shortcoming below).

We proceed by first of all investigating the basic distinctions and insights present within the mathematical thought of Paul Bernays.

2. Preliminary background remarks

Emerging from various ancient civilizations the discipline of mathematics appeared to be appreciated as one of the prime examples of *human rationality*. During the era of Greek antiquity significant advances were made, particularly in respect of deductive reasoning, exemplified in Euclid's "Elements." Of course this legacy was still largely in the grip of the Pythagorean conviction that the *essence* of everything could be captured by natural numbers and their ratios (fractions).

At the same time this restriction to rational numbers (fractions) prepared the way towards what eventually became known as the first *foundational crisis* of mathematics – given in the discovery of the irrational numbers. The inability to conquer irrational numbers in an arithmetical way, merely employing fractions, resulted in the geometrization of mathematics. For this reason Euclid treated the theory of number as a *part of geometry*.[3]

The initial intellectual stimulus setting mathematics on its path of development therefore derives from the atomistic Pythagorean thesis that *everything is number*. However, this approach soon had to revert to a spatial perspective, in terms of which

3 "Das begründet einen Vorrang der Geometrie vor der Arithmetik, und die Konsequenz sind die Bücher des Euklid: Die Theorie der Zahlen ist ein Teil der Geometrie" – Laugwitz, 1986:10.

the importance of the (spatial) *whole-parts relation*[4] dominated the subsequent development of mathematics up to the 19th century. Eudoxos has already approximated the discovery of the modern calculus (later independently attributed to Leibniz and Newton), but restricting himself to the perspective of space prevented him from contemplating the *arithmetical* concept of a limit, which eventually turned out to be an indispensible starting-point for the development of the calculus and mathematical analysis.[5] Having resolved the continuum into a set of (isolated) points, set theory then super-imposes upon it, with the auxilliary set theoretical construction of environments and open sets, a "topology" within which it is once again possible to speak of "continuity" ("Stetigkeit") (see Laugwitz, 1997:266). Laugwitz also mentions that Cantorean set theory advances an "atomistic conception" of "the continuum." The position of intuitionism remained ambiguous because Brouwer's "basal intuition" embraced both the elements of *discreteness* and *continuity*.[6]

3. Key distinctions in the mathematical thought of Paul Bernays

Bernays wrestled with all these problems and advanced a distinct view on the issues involved in the claim, made by Cantor (1845-1918) and most of his followers, namely that mathematics can be fully *arithemetized*.

His position entails a stance in which one finds clear distinctions regarding

1) what is universal and what is individual;
2) what is discrete and continuous (including the infinite divisibility of the latter)
3) what is given as an extended whole with intrinsic positions but not as an ordered multiplicity of points; and
4) that the idea of the continuum originally is *geometrical* and that mathematical analysis expresses it in *arithmetical terms*;
5) that what belongs to human reflection and what is "ontically" given[7] ought to be distinguished, based upon the assumption
6) that that there are "objective" quantitative properties in reality;
7) that spatial continuity is irreducible while expressing
8) the *totality character* of continuity; and
9) that therefore the continuum cannot be fully arithmetized;
10) Finally, he also displayed an insight into the difference between *mathematical space* and *physical space*, and also employed the distinction
11) between a law and what is subjected to such a law and

4 Without recognizaing its connection with the aspect of space Russell later on acknowledged the whole-parts relation as basic: "The relation of whole and part is, it would seem, an indefinable and ultimate relation" (Russell, 1956:138).

5 In general a number l is called the limit of the sequence (x_n), when for an arbitrary $0 < \in$ a natural number n_0 exists such that $|x_n - l| < \in$ for all $n \geq n_0$.

6 Brouwer states: "Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear contniuum, i.e., of the 'between,' which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units" (Brouwer, 1964:69).

7 In sub-paragraph 3.4 below a more extensive account of the meaning of the expression "ontically given" is found – with reference to what Gödel called "an aspect of objective reality" and what Bernays designated as "conforming to the nature of the matter (*naturgemäß*)." The aspects of reality are not the product of human thinking since they *make possible* – in an ontic sense – human experience of them and human reflection upon them. Obtaining knowledge (the *epistemic perspective*) therefore differs from what is ontically given (the ontological issue).

12) between finite and infinite totalities.

Particularly within the intellectual atmosphere of mathematicians who were in the grip of an arithmeticistic orientation, the stance of Bernays is truly remarkable. Let us explore his position in more detail.

3.1 Defining mathematics

Defining mathematics may appear, at first sight, to be straight-forward. Two considerations immediately surface: (i) any definition of mathematics exceeds the boundaries of mathematics, and (ii) any definition becomes absurd when it does not "touch reality."

Bernays considers the concept of an *ordinal number* as a basic element of mathematics (Bernays, 1976:41), although his definition of mathematics goes further. He does not want to proceed on the basis of the opposition between what is qualitative and quantitative, In a phenomenological sense the qualitative is opposed to the *structural*. As far as mathematics is concerned this structural trait exists in the forms of next-to-each-other, after-each-other, and being composed, accompanied by all the conceptual determinations and law-conformities related to them (Bernays, 1976:141). Of course one may be tempted to point out that within widely differing contexts *structures* are encountered (Bernays mentions structures of society, economic structures, the structure of the earth, of plants, and so on). For this reason Bernays provides a more precise qualification of what we would call *modal abstraction* when he refers to *possible structures*, in particular the "idealisierten strukturen" (*idealized structures*) (Bernays, 1976:172).

The problem with this definition is unveiled through the question: are these structures given or are they constructed by the mathematician? From the above-mentioned explanation it may appear as if the focus is on *constructed structures*. Yet in a different context Bernays argues that the "Gegenstand" (object) of scientific investigation must be given prior to (*vorgängig*) this research (Bernays, 1976:110). On the basis of what is thus given, mathematics further explores the development of additional theoretically articulated elements (Bernays, 1976:110). Clearly, his general definition, namely: "Mathematics is concerned with possible structures, and indeed in particular idealized structures" (Bernays, 1976:172), intends to account for the interplay between what is given and what is developed on that basis. One may also say: what is *disclosed* on the basis of what is *given*. Bernays opts for the view that certain ontic traits of reality enable the formulation of the axioms of arithmetic. Hao Wang remarks that Kurt Gödel was very "fond of an observation that he attributes to Bernays": "That the flower has five petals is as much part of objective reality as that its color is red" (quoted by Wang, 1982:202). In other words, "objective reality displays a numerical aspect."

Because every response to what is given in an ontic sense is historically "dated," every attempt to define mathematics exclusively in such "response terms" will invalidate and actually eliminate the history of mathematics altogether. For example, saying that mathematics *is set theory* eliminates all forms of mathematics before Cantor introduced his set theory. Hersh is therefore justified in his questioning of this view: "What does this assumption, that all mathematics is fundamentally set theory, do to Euclid, Archimedes, Newton, Leibniz, and Euler? No one dares to say they were thinking in terms of sets, hundreds of years before the set-theoretic reduction was invented" (Hersh, 1997:27). This remark underscores our statement above, namely that any definition of mathematics becomes absurd when it does not "touch reality" [point (ii)].

Touching reality makes an appeal to a point of departure that is "ontically given" – see footnote 7 above.

The best way to articulate the nature of mathematics in this regard is to accept what is *ontically given*, namely, among other, the aspects of number and space, and then account for the possibility to explore (investigate) the meaning of these aspects (and their interrelations) by means of our theoretical reflection on them – resulting in the historical development of the discipline of mathematics. This is precisely the point of view defended by Bernays. According to him an operative conception of mathematics will hold that mathematics certainly has to bring forth its own objects. Alternatively, he suggests, one may acknowledge that the *Gegenstand* [object] of mathematics is something given to us prior to our reflection and that through the concepts we form and by means of our axiomatic descriptions we open up and make what is given accessible to human cognition.[8]

The first above-mentioned consideration stated that any definition of mathematics exceeds the boundaries of mathematics. For example, when it is said that "mathematics consists of algebra and topology" then it is obvious that this definition itself is not an axiom or theorem either within algebra or within topology. Posing and answering this question belongs to the philosophical foundation of mathematics. The issue is not who provides the definition but *what is the nature of this definition*. Mathematicians may want to argue that only a mathematician can tell us what mathematics is. However, without denying any mathematician the right to answer it, the answer will still not be *mathematical* in nature!

If there is something *given* prior to the practice of mathematics, then the next question is: how do we obtain knowledge of this given ontic reality?

3.2 How do we obtain knowledge of ontic reality?

Bernays answers that "idealizations are characteristic for the formation of mathematical concepts" (Bernays, 1976:VIII). The limitations of our empirical world, restricting us to what is finite, can only be exceeded by what Bernays calls "the formal abstraction which helps us to transcend the boundaries of factuality" (Bernays, 1976:38). As soon as one contemplates the infinite divisibility of space, it becomes clear that our metrical spatial representations become meaningless in a physical sense (Bernays, 1976:38). Likewise, the foundational assumption of *infinite totalities*, employed in analysis, cannot be grasped in our intuition but only by means the formation of ideas (*Ideenbildung*). Although Bernays does not know the idea of modal (functional) aspects of reality, his account of discreteness and continuity is completely equivalent to the aspects of *number* and *space* in Dooyeweerd's theory of modal aspects.

3.3 "Arithmetic" and the "geometric" intuitions

What, according to Bernays, is characteristic of our distinction between the "arithmetic" and the "geometric" intuitions? He rejects the widespread view that this distinction concerns *time* and sp*a*ce, for according to him, the proper distinction needed is

8 "Eine operative Auffassung der Mathematik wird von vielen verfochten. Für diese ist characteristisch, daß sie den Gegenstand der Mathematik nicht in etwas vorgängig Vorliegendem erblickt, das durch die Begriffsbildungen und axiomatischen Beschreibungen für unser Erkennen zugänglich gemacht werden soll, sondern das mathematische Operieren selbst und the Gegenständlichkeiten, die darin zustande kommen, als das Thema der Mathematik ansieht. Die Mathematik soll hiernach ihre Gegenstände gewissermaßen selbst erzeugen" (Bernays, 1976:114).

that between the *discrete* and the *continuous*.[9] This characterization on the one hand supports the way in which Fraenkel et al. portrays the history of mathematics and on the other coincides with Dooyeweerd's view.

Fraenkel et al. address this issue in their 1973 work on the *Foundations of Set Theory*: "Bridging the gap between the domains of discreteness and of continuity, or between arithmetic and geometry, is a central, presumably even the central problem of the foundation of mathematics" (Fraenkel, A., et al., 1973:211).[10] We may quote other scholars, having the same view: "from the earliest times two opposing tendencies, sometimes helping one another, have governed the whole involved development of mathematics. Roughly these are the discrete and the continuous" (Bell, 1965:12); and Rucker remarks: "The discrete and continuous represent fundamentally different aspects of the mathematical universe" (Rucker, 1982:243).

But then the question recurs: what is the relationship between the "discrete" and the "continuous"? Bernays, in an unequivocal way, declares that the representation of number is more basic than that of space (Bernays, 1976:69).[11] By contrast, since everything continuously extended can be comprehended at once, Greek mathematics believed that space is more fundamental than number (Fraenkel, et al., 1973:213). Also modern mathematicians adhered to the same view, in particular Gottlob Frege and René Thom.

According to Frege, devastated by the discovery of the inconsistency of naive set theory by Russell and Zermelo in 1901, mathematics ultimately is geometry (Frege, 1979:277). Longo mentions that René Thom also upholds the view that continuity *precedes* discreteness: "For him, as for many mathematicians of the continuum, 'the Continuum precedes ontologically the discrete', for the latter is merely an 'accident coming out of the continuum background', 'a broken line' " (Longo, 2001:6).[12]

According to Bernays however, the number concept is more immediate to our understanding than the representation of space.[13] In this respect Bernays partially conforms to the growing view of modern mathematics, also found in the thought of Frege. In 1884 Frege asserted the view that reaffirms the newly emerging arithmeticism during the last couple of decades of the 19th century. He asked the qustion: "Is it not the case that the basis of arithmetic is deeper than all our experiential knowledge and even deeper than that of geometry?"[14] However, as we shall see below, Bernays, similar to Frege, ultimately did not adhere to an *arithmeticistic* orientation.

9 "Es empfiehlt sich, die Unterscheidung von 'arithmetischer' und 'geometrischer' Anschauung nicht nach den Momenten des Räumlichen und Zeitlichen, sondern im Hinblick auf den Unterschied des Diskreten und Kontinuierlichen vorzunehmen" (Bernays, 1976:81).

10 It must be kept in mind that contemporary scholars currently still refer to this work as authoritative (see for example the references in Maddy, 1997:14, 39-42, 47-48, 50, 52-54, 57-58, 61, 85).

11 "Die Vorstellung der Zahl ist elementarer als die geometrischen Vorstellungen."

12 Later on in this article Longo combined Thom's views with those of Leibniz: "By contrast Leibniz and Thom considers the continuum as the original giving, central to all mathematical construction, while the discrete is only represented as a singularity, as a catastrophe" (Longo 2001:19).

13 "Der Zahlbegriff ist für unsern Geist unmittelbarer als die Vorstellung des Raumes" (Bernays, 1976:75).

14 "Liegt nicht der Grund der Arithmetik tiefer als der alles Erfahrungswissens, tiefer selbst als der der Geometrie?" (Frege, 1884:44). In passing we may note that Tait is justified in his criticism of Frege's confusion in his understanding of the meaning of the word 'gleich': "His reading seems to me to have been misdirected by two related things: his interpretation of 'gleich' to mean 'identical' and his failure to understand the historical use of the term 'number' to mean what is numbered" (Tait, 2005:242).

3.4 Arithmetic and Logic

Closely related to the distinction between discretenes and continuity, while accepting the foundational role of discreteness, another issue surfaces, namely the question what the relationship between logic and arithmetic is?

Bernays points out that according to Hilbert there is a circularity entailed in the logicist attempt to deduce the quantitative meaning of number from that of the logical-analytical mode. He quotes Hilbert saying:

> Only when we analyze attentively do we realize that in presenting the laws of logic we already have had to employ certain arithmetical basic concepts, for example the concept of a set and partially also the concept of number, particularly as cardinal number [Anzahl]. Here we end up in a vicious circle and in order to avoid paradoxes it is necessary to come to a partially simultaneous development of the laws of logic and arithmetic (Bernays, 1970:199).

A similar idea is found in the thought of the well-known neo-Kantian thinker, Ernst Cassirer, who taught philosophy to Bernays. He acknowledged *original functions* (aspects) that are irreducible ("not in need of genuine derivation").[15] He therefore had an understanding of "original functions" [Urfunktionen] and their mutual relationships, accounting for the similarities and differences between them. He distinguishes between "a logical identity and diversity and a numerical unity and difference."[16]

Bernays is convinced that mathematical knowledge represents a higher abstraction than what is found within logic. "As we have established, in respect of the formal, the mathematical perspective in opposition to the conceptual logical represents the standpoint of a higher abstraction."[17]

Bernays was convinced that the number concept is original and non-logical in nature: "What is neessary is to acknowledge that deductive logic already includes intuitive evidence and that the logical number definitions do not show that for example the number concepts as such are specifically logical in nature (as pure reflective concepts), for they rather are solely normings of elementary structural concepts."[18] What is normally designated as "formal elements" actually represent the logical side of statements, although Bernays repeatedly affirms that the logical terms and principles partially relate to certain very general characteristics of reality ["gewisse sehr allgemeine

15 "In order not to accept a regressus in infinitum a critical analysis of knowledge has to stop at specific original functions which are not in need of genuine derivation and which are also not capable of it" ["Denn die kritische Analyse der Erkenntnis wird, wenn man nicht einen regressus in infinitum annehmen will, immer bei gewissen Urfunktionen Halt machen müssen, die einer eigentlichen 'Ableitung' weder fähig noch bedurftig sind"] (Cassirer, 1957:73).

16 "In der Tat ist nicht einzusehen, warum man lediglich logische Identität und Verschiedenheit, die als notwendige Momente in den Mengenbegriff eingehen, als solche Urfunktionen gelten lassen und nicht auch die numerische Einheit und den numersichen Unterschied von Anfang an in diesen Kreis aufnehmen will. Eine wirklich befriedigende Herleitung des einen aus dem anderen ist auch der mengentheoretischen Auffassung nicht gelungen, und der Verdacht eines versteckten erkenntnistheoretischen Zirkels blieb gegenüber allen Versuchen, die in dieser Richtung gemacht werden, immer bestehen" (Cassirer, 1957:73-74).

17 "In hinsicht auf das Formale stellt aber, wie wir fanden, die mathematische Betrachtung gegenüber der begrifflich logischen den Standpunkt der höheren Abstraktion dar" (Bernays, 1976:27).

18 "Dort kam es darauf an, zu erkennen, daß in die deduktive Logik bereits anschauliche Evidenz eingeht, und daß die logischen Anzahl-Definitionen nicht etwa die Anzahlbegriffe als solche von spezifisch logischer Natur (als reine Reflexionsbegriffe) erwiesen, sondern vielmehr nur logische Normierungen elementarer Strukturbegriffe sind" (Bernays, 1976:46).

Chracteristica der Wirklichkeit"]. It implies that both the logic of our everyday language and symbolic logic contain at once, adjacent to each other, elements that are formally and object-wise ["*gegenstänlich*"] motivated.[19]

The possibility to grasp arithmetical and in particular numerical propositions in logical terms may appear to justify the traditional view that logic represents the more general perspective. However, Bernays does not hesitate to affirm that notwithstanding this possibility arithmetic is still the more general ("purer") schema, showing upon closer investigation that it is unjustified to see logical universality as the highest universality.[20]

What is therefore at stake in this regard, is the acknowledgement that concept formation and definition ultimately rests upon the acceptance and employment of primitive terms. In order to avoid a *regressus in infinitum*, this state of affairs ought to be respected. Cassirer has a clear understanding of this when he writes:

> In order not to accept a regressus in infinitum a critical analysis of knowledge has to stop at specific original functions which are not in need of genuine derivation and which are also not capable of it (Cassirer, 1957:73).

3.5 Ontic conditions

Not only do we have to distinguish between number and space (discretenes and continuity) but also between these two and the logical-analytical aspect. Moreover, the key terms involved in rational conceptual understanding are themselves not open to (rational) conceptual definition, for they are, as Cassirer puts it, *Urfunktionen* (original functions)![21] These ontic conditions not only make possible our concept of numbers but also explain why Bernays rejects the idea that an axiomatic system in its entirety is an arbitrary construction: "One cannot justifiably object to this axiomatic procedure with the accusation that it is arbitrary since in the case of the foundations of systematic arithmetic we are not concerned with an axiom system configured at will for the need of it, but with a systematic extrapolation of elementary number theory conforming to the nature of the matter (*naturgemäß*)."[22] The "nature of the matter" contains an implicit reference to the *ontic status* of the "multiplicity-aspect" of reality and it presupposes an awareness of the difference between the various (modal, functional) aspects of reality and the conrete dimension of entities and events functioning within these aspects. Sometimes the distinction between aspects and entities is captured by referring to the difference between modal laws and type laws (modality and typicality). Natural and social entities function in a "typical" way within every modal aspect. The word "typical" actually refers to the *typonomic* specification of entitary functions (*typos* =

19 "In der Logik, und zwar sowohl in derjenigen der Umgansprache wie in der symbolischen Logic, haben wir nebeneinander formal und gegenständlich motivierte Elemente. Eine gegenständliche Motivierung liegt insofern vor, als die logischen Termini und Prinzipien zu einem Teil Bezug haben auf gewisse sehr allgemeine Characteristika der Wirklichkeit" (Bernays, 1976:80).

20 "Ungeachtet also der Möglichkeit der Einordnung der Arithmetik in die Logistik stellt die Arithmetik das abstraktere ('reinere') Schema dar, und dieses erscheint als paradox nur auf Grund einer traditionellen, aber bei näherem Zusehen nicht gerechtfertigten Ansicht, wonach die Allgemeinheit des Logischen in jeder Hinsicht die höchste Allgemeinheit bilbet" (Bernays, 1976:135).

21 Mühlenberg points out that already for Aristotle thinking presupposed knowledge that was not mediated by any proof (Mühlenberg, 1966:73).

22 "Gegen diese axiomatische Vorgehen besteht auch nicht etwa der Vorwurf der Willkürlichkeit zu Recht, denn wir haben es bei den Grundlagen der systematische Arithmetik nicht mit einem beliebigen, nach Bedarf zusammengestellten Axiomensystem zu tun, sondern mit einer naturgemäßen systematischen Extrapolation der Elementare Zahlenlehre" (Bernays, 1976:45).

type and *nomos* = *law*). Therefore *typical functions* can also be designated as *typonomic functions*. Whereas modal laws are normally discerned through *modal abstraction*, the discovery and analysis of type laws are dependent (at least in physics) upon experimentation.

According to Bernays there are two kinds of factuality: (i) *modal subjects* (such as *numbers* and *spatial figures* that are factually subjected to their correspondig numerical and spatial laws); and (ii) *typical subjects* (factual entities, such as atoms and molecules, material things, plants, animals and humans). Bernays touches upon these differences when he refers to the *intuitive,* the *theoretical* and the *experimental*.[23] In his discussion of Wittgenstein it is noteworthy that Bernays rejects the view of those who merely acknowledge one kind of factuality, that which is concrete: "It appears that only a pre-conceived philosophical view determines this requirement, that view namely, according to which there can solely exist *one* kind of factuality, that of concrete reality."[24]

Bernays also appreciates the similarities and differences between different kinds of space, owing to the fact that it became necessary to distinguish between physical space and mathematical space determined by geometrical laws: "It was only through the recent development of geometry and physics that it appears to be necessary to distinguish between space as something physical and space as an ideal multiplicity determined by geometrical laws" (Bernays, 1976:37).[25]

3.6 Analogical interconnections

Having proposed a clear distinction between discreteness and continuity one may contemplate the question: are there for Bernays *elements of coherence* between these two domains?

First of all it must be mentioned that Bernays repeatedly emphasizes that the idea of the continuum originally is a *geometrical idea*. The immediate implication is that for him the reduction of continuity to what is discrete is only successfull in an approximative sense.[26] For this reason he declares that a complete arithmetization of the idea of the continuum cannot be justified because this idea in the first place (*ursprünglich*) is a geometrical idea.[27] The word *urpsrünglich* reflects the *original meaning* of continuity, suggesting that in different or other contexts one can merely encounter a non-original, that is to say, an *analogical* employment of the idea of continuity. Bernays indeed does not hesitate to prusue this path, for he holds that in the mathematical discipline of *analysis* the idea of the continuum is expressed in an arithmetical language: "The idea of the continuum is a geometrical idea, which in analysis is expressed in an arithmetical language."[28]

23 "dem intuitive, dem theoretischen und dem experimentellen" (Bernays, 1976:108).

24 "Es scheint, daß nur eine vorgefaste philosophische Ansicht dieses Erfordernis bestimmt, die Ansicht nämlich, daß es nur *eine* Art von Tatsächlichkeit geben könne, diejenige der konkreten Wirklichkeit" (Bernays, 1976:122).

25 He does not hesitate to speak of the "permanence of laws" (Bernays, 1976:75).

26 "Tatsächlich gelingt auch die Reduktion des Stetigen auf das Diskrete nur in einem angenäherten Sinn" (Bernays, 1976:82).

27 "Jedoch, es ist sehr zweifelhaft, ob eine restlose Arithmetisierung der Idee des Kontinuums voll gerecht werden kann. Die Idee des Kontinuums ist, jedenfalls ursprünglich, eine geometrische Idee" (Bernays, 1976:188).

28 "Die Idee des Kontiuums ist eine geometrische Idee, welche durch die Analysis in arithmetischer Sprache ausgedrückt wird" (Bernays, 1976:74).

The way in which the arithmetical perspective analogically reflects the original meaning of continuity is by acknowledging that the completeness [totality character] it evinces ("jenen Character der Geschlossenheit") stands in the way of a complete arithmetization of the continuum: "It derives from the fact that the intuitionistic representation does not display that characteristic completeness which undoubtedly belongs to the geometrical representation of the continuum. And it is this feature which obstructs a perfect arithmetization of the continuum" (Bernays, 1976:74).[29]

3.7 Potential infinity and actual infinity

Since Aristotle philosophy and mathematics employed these two phrases to distinguish between two kinds of infinity. Apart from the rich history attached to them we are currently merely interested in the fact that as soon as the difference between them is at stake the notion of a *totality* surfaces. Even Hilbert himself speaks of the "totality of the numbers 1, 2, 3, 4, ..." when he explains the nature of the actual infinite. Lorenzen, who studied with Hilbert as a school boy, eventually opted for a constructive mathematics in which the actual infinite is questioned. Yet, owing to his thorough understanding of the dominating rol of infinity in modern mathematics he acknowledges that an account of the real numbers using the actual infinite reveals its ties with space and geometry:

> The overwhelming appearance of the actual infinite in modern mathematics is therefore only understandable if one includes geometry in one's treatment. ... The actual infinite contained in the modern concept of real numbers still reveals that it originates from geometry (*Herkunft* – Lorenzen, 1968:97).

From the above-mentioned remarks of Bernays it is particularly the *totality character* of continuity that caused him to point out that the idea of the continuum originally is a geometrical idea. Reflecting on the same assumption Lorenzen focuses on another feature of spatial continuity, namely that it is determined by an order of simultaneity (at once). This is seen when he explains how one can account for real numbers in terms of the actual infinite:

> One imagines much rather the real numbers as all at once actually present – even every real number is thus represented as an infinite decimal fraction, as if the infinitely many figures (*Ziffern*) existed all at once (*alle auf einmal existierten*) (1972:163).[30]

The mode of speech employed by Lorenzen, similar to that of Bernays, makes it plain that arithmetic by itself does not provide any motive for the introduction of the actual infinite, as correctly pointed out by Lorenzen (Lorenzen, 1972:159). From a different corner Körner also holds the view that the basic difference between arithmetic and

29 "Das rührt davon her, daß die intuitionistische Vorstellung nicht jenen Charakter der Geschlossenheit besizt, der zweifellos zur geometrischen Vorstellung des Kontinuums gehört. Und es ist auch dieser Charakter, der einer vollkommenen Arithmetisierung entgegensteht."

30 Bernays reiterates a long-standing practice, dating back to Plotinus, Augustine, and Maimon, when he refers to the thought of divine omniscience which should be attributed with the capacity to overlook an infinite totality in one purview (analogous to Lorenzen's "*alle auf einmal*"): "Wenn wir schon den Gedanken einer göttlichen Allwissenheit konzipieren, so würden wir dieser doch zuschreiben, das sie eine Gesamtheit, deren jedes einzelne Element uns grundsätzlich zugänglich ist, in einem Blick überschaut" (Bernays, 1976:131).

analysis in its classical form is that the latter defines real numbers with the aid of actual infinite totalities.[31] ("aktual unendlicher Gesamtheiten" – Körner, 1972:134).

Sometimes Bernays uses an alternative mode of speech. Infinite multiplicities (*Unendliche Mannigfaltigkeiten*) provides access to our *thinking* only (Bernays, 1976:39). The postulates of analysis cannot be verified in intuition and the same applies to "infinite totalities" which can solely be grasped by the formation of ideas (*Ideenbildung*) (Bernays, 1976:44).

4. Points of connection with the thought of Dooyeweerd and Gödel

Dooyeweerd is primarily known for his theory of modal aspects, although it must be remembered that this theory is constitutive for his theory of "individuality structures," aimed at accounting for the structural laws or principles holding for multi-aspectual (natural and social) entities. The important assumption of the theory of modal aspects is that they are not mere modes of thought for he holds that they are *ontic a priori's*, truly existing in a transcendental sense, that is, in the sense of partially *making possible* whatever functions within them (natural and social entities and concrete processes). Dooyeweerd therefore, just like Bernays, also distinguishes between "two kinds of factuality." Modal or typical facts are always correlated with modal laws and typical laws. Within the modal aspects the mature conception of Dooyeweerd distinguishes between the law-side and the factual side of each aspect and on the factual side he distinguishes between subject-subject relations and subject-object relations.

Gödel, in his own way, struggled with the ontic status of numerical relationships, referred to by him as "objective reality." His argument in support of the acknowledgement of "objective aspects of reality" proceeds from the idea of "semiperceptions." Distinct from physical data Gödel argues that "mathematical objects" can be accessed through this second kind of perceptions, namely "semiperceptions." Obviously data of this second kind "cannot be associated with actions of certain things upon our sense organs" (quoted by Wang, 1988:304). In terms of the idea of *ontic aspects* these "semiperceptions" indeed relate to the functional aspects of reality. Gödel says:

> It by no means follows, however, [that they] are something purely subjective as Kant says. Rather they, too, may represent "an aspect of objective reality" (my emphasis – DS), but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality (quoted by Wang, 1988:304).

Although Wang is inclined to agree with Gödel he does "not know how to elaborate his assertions" (Wang, 1988:304). The way in which Wang expresses himself is simply a sign of lacking the distinction between the dimensions of (natural and social) entities and modal aspects. In terms of this distinction one can simply say that the various aspects (including the numerical) belong to "objective" reality and therefore display an ontic status. Yet they do not concern the concrete "whatness" of the dimension of entities, for they embody the "howness" of the functional modes of real-

31 "Dieser grundlegende Unterschied zwischen elementarer Arithmetik und Analysis in ihrer klassischen Form beruht auf der Tatsache, daß der zentrale Begriff der Analysis, der einer reellen Zahl, mit Hilfe aktual unendlicher Gesamtheiten definiert wird" (Körner, 1972:134).

ity. The aspects as *functional modes of being* are just as "real" as the any concretely existing entities "out there."

Dooyeweerd, Gödel and Wang indeed advance the idea that "reality" also embraces the "ontic" "aspects of reality," designated by Gödel and Wang as "objective." According to Gödel these *aspects* are not like "concrete entities" occupying "a location in spacetime." We must note that the above-mentioned philosopher, Ernst Cassirer, effectively captured the legacy of Western philosophical thinking in the title of a significant book, *Function Concept and Substance Concept*, directed at the distinction between function and entity ("substance"). The first edition of this book appeared in 1910, and it covers both the Greek-Medieval and the modern (post-Renaissance) eras. The former was pre-occupied with concrete entities – while the latter increasingly started to appreciate *functional relations* within reality.

Whereas traditional Greek and medieval philosophy tend to be *substantialitstic*, modern philosophy by and large opted for a *functionalistic* stance. Dooyeweerd wanted to avoid both these extremes. According to him the possibility to identify different aspects is guaranteed by their respective *meaning-nuclei*. The meaning-nucleus of an aspect secures its uniqueness, its irreducibility and this core meaning is *indefinable* ("primitive"). Owing to the relativity of lingual meanings the constant challenge to philosophy and the special sciences is to ensure that the lingual terms employed to designate their core meanings are appropriate. What is remarkable in this connection is that Dooyeweerd‘s designation of the core meaning of the aspects of number and space is coinciding with the above-mentioned proposal of Bernays: *discreteness* and *continuity*!

In addition both Dooyeweerd and Bernays held the view that number is more basic than space. Dooyeweerd would prefer to say that the numerical aspect is *foundational* to the spatial aspect. This entails his idea of the *interconnectedness* between the various aspects, exemplified in backward and forward pointing *analogies* between them. The former are also known as *retrocipations* and the latter as *anticipations*. Since the numerical aspect is not preceeded by any aspect, it does not have any retrocipatory analogies. The spatial aspect, however, since it is founded in the numerical mode, does have retrocipatory analogies to number. On the law side of space Dooyeweerd discerns *dimensionality* as a numerical analogy (1, 2, 3 or more dimensions) and on the factual side *magnitude* is identified as an arithmetical analogy within the aspect of space (Dooyeweerd, 1997-II:85 ff.).

Dooyeweerd's theory of modal aspects and the analogical moments which reflect their unbreakable coherence represents a particular answer to one of the classical problems of philosophy, namely how to account for *uniqnueness* and *coherence* (the *coherence of irreducibles*). The specific answer given by Dooyeweerd to this problem introduced new descriptive terms: he captures the uniquness of modal aspects with reference to their *sphere-sovereignty* while their mutual coherence (through retrocipations and anticipations) is depicted as their *sphere-universality*. What guarantees the sphere-sovereignty of an aspect is its indefinable, primitive meaning-nucleus.

The acknowledgement of "primitive terms" highlights the first element of the just mentioned problem regarding the *coherence of irreducibles*. What Gödel had to say in this regard approximates closely what Dooyeweerd said. Yourgrau explains that Gödel "insisted that to know the primitive concepts, one must not only understand their relationships to the other primitives but must grasp them on their own, by a kind of 'intuition' " (Yourgrau, 2005:169). Dooyeweerd also makes an appeal to an immediate, intuitive insight into the core meaning of an aspect which cannot be captured in a conceptual definition (see Dooyeweerd, 1997-II:475). When Gödel continues on the next page by stating that "the fundamental concepts are primitive and their meaning is not exhausted by their relationships to other concepts," this formulation is as close one can get to the (modal) principles of sphere-sovereignty and sphere-universality!

However, Gödel's acknowledgement of what cannot be defined also encompasses the basic notion of a *set*:

> The operation 'set of x's' (where the variable 'x' ranges over some given kind of objects) cannot be defined satisfactorily (at least not in the present state of knowledge), but can only be paraphrased by other expressions involving again the concept of set, such as: 'multitude of x's', 'combination of any number of x's', 'part of the totality of x's', where a 'multitude' ('combination', 'part') is conceived as something that exists in itself, no matter whether we can define it in a finite number of words (so that random sets are not excluded) (Gödel, 1964:262).

Yet Gödel did not develop a theory of modal aspects as such, which explains why he did not "position" primitives in the sense of indefinable meaning-nuclei within modal aspects in their qualifying role in respect of all the analogical structural elements found within any one of them. It is not the primitives themselves (the meaning-nuclei) that point backward and forward (retrocipate and anticipate), but the various aspects.

The astonishing impact which Gödel's 1931 publication had on our understanding of mathematics and logic may acquire a deepened perspective when it is seen in relation to the emphasis Dooyeweerd from very early on laid upon the self-insufficiency of human thought. The mere idea that the irreducible core meaning of an aspect is indefinable (*primitive*) and can solely be approximated by means of immediate, intuitive insight, underscores the limits of conceptual rationality. Ultimately rational analysis and concept-formation is therefore dependent upon terms exceeding a rational conceptual grasp. This outcome is similar to the outcome of Gödel's result that any consistency proof of an axiomatic system necessarily exceeds the formalism of the system.

Grünfeld explains Gödel's achievement as follows:

> Gödel proved that if any formal theory T that is adequate to include the theory of whole numbers is consistent, then T is incomplete. This means that there is a meaningful statement of number theory S, such that neither S nor not-S is provable within the theory. Now either S or not-S is true; there is then a true statement of number theory which is not provable and so not decidable. The price of consistency is incompleteness (Grünfeld, 1983:45 – see also Hofstadter, 1980:86-87).

Recollecting the optimistic closing lines of Hilbert's 1930 speech at the occasion of receiving honorary citizenship of Königsberg, namely "Wir müssen wissen, Wir werden wissen," may help us to realize what a big blow Gödel's finding was for this expectation, that is, for the hope of *proving* the consistency of mathematics. Surely Hilbert's own brilliant student, Hermann Weyl (1885-1955), who left his (axiomatic-)formalist orientation in favour of Brouwer's intuitionism, has had a thorough understanding of the predicament in which Hilbert found himself. Weyl comments strikingly in this regard: "It must have been hard on Hilbert, the axiomatist, to acknowledge that the insight of consistency is rather to be attained by intuitive reasoning which is based on evidence and not on axioms" (Weyl, 1970:269).

5. The actual infinite

The larger part of the history of philosophy and mathematics accepted only the potential infinite and advanced a negative assessment of the *actual infinite*. Since and after Gregory of Nyssa (335-394^{+}) actual infinity was ascribed to God, opposed to the finiteness of the world. Nicholas of Cusa (1401-1464) altered this restriction by introducing the idea that whereas the world is potentially infinite, God is actually infinite. Spinoza (1632-1677) completed the circle later on in seeing both the world and God as actually infinite, a view revived in the thought of Cantor.

The latter is renowned for his employment of the actual infinite in mathematics, thus transcending the objection of Gauss 1777-1855), who said in a letter to Schumacher in 1831 that within mathematics it is never allowed to see the infinite as something completed, and Kronecker (1823-1891), who aimed at restricting mathematics to the potential infinite. However, at this point the views of Dooyeweerd and Bernays part ways. Dooyeweerd appears to follow the intuitionism of Brouwer (1881-1966) and Herman Weyl, while Bernays advanced an "as if" approach towards the actual infinite.[32]

In spite of his mentioned well-articulated account of the sphere-sovereignty of the aspects of number and space, guaranteed by their respective meaning-nuclei, *discreteness* and *continuity*, Dooyeweerd accepted infinity only as something unfinished. According to him the infinite is merely a *law* determining an *endless succession*. It is on the basis of this restricted criterion that he criticizes the transfinite arithmetic of Cantor as well as the so-called *antinomies* of the actual infinite (Dooyeweerd, 1997-II:87).[33] The intuitionistic mathematics of Brouwer and Weyl surely did influence Dooyeweerd in his rejection of the actual infinite in mathematics, as it is manifest in his view that an infinite succession of numbers is determined "by the law of arithmetical progression" making it possible "to determine the discrete arithmetical time of any possible finite numerical relation in the series" (Dooyeweerd, 1997-II:92). The same assumption underlies his following remark: "Numbers and spatial figures are subject to their proper laws, and they may not be identified with or reduced to the latter. This distinction is the subject of the famous problem concerning the socalled 'actual infinity' in pure mathematics. The principle of progression is a mathematical law which holds good for an infinite series of numbers or spatial figures. But the infinite itself cannot be made into an actual number" (Dooyeweerd, 1997-I:98-99, note 1). Dooyeweerd's dependence is explicit when, with reference to the "speculative-constructive basis" of Cantor's theory of transfinite numbers, he refers to Weyl (1931) (Dooyeweerd, 1997-II:340, note 1).[34]

Being the co-worker of the leading mathematician of the 20th century, David Hilbert, it is understandable that Bernays supported the employment of the actual infinite within mathematics. Hilbert after all said: "No one shall drive us out of the para-

32 Recently restricting infinity to the potential infinite (successive infinite) once more surfaced in the argument of Karl-Heinz Wolff against the non-denumarability conclusion of Cantor's well-known diagonal proof (see Wolff, 2010).

33 In passing we may note that another legal scholar, Felix Kaufmann, dedicated a whole work in support of his rejection of the actual infinite within mathematics – see Kaufmann, 1930.

34 Dooyeweerd did not know about the development of "lawless sequences" in intuitionism, especially after Brouwer's article of 1952 – see Troelstra and van Dalen 1988 and Kreisel 1968.

dise which Cantor has created for us" (Hilbert 1925:170).[35] Yet the justification which Bernays gave for the use of actual infinity represents a unique position, as mentioned, an "as if" approach which is indeed worth considering.

Vaihinger initially developed a whole philosophy of the "as if" (*Die Philosophie des Als Ob*), in which he attempts to demonstrate that various academic disciplines may positively use certain fictions which in themselves are intrinsically antinomic. As examples of "fruitful" fictions he mentions the infinitely large and the infinitely small (cf. Vaihinger, 1922:87 ff., 530). Ludwig Fischer presents a more elaborate mathematical explanation of this notion of a fiction. In general he argues: "The definition of an irrational number by means of a formation rule always involves an 'endless', i.e. unfinished process. Supposing that the number is thus given, then one has to think of it as the completion (Vollendung) of this unfinished process. Only in this ... the internally antinomic (*in sich widerspruchsvolle*) and fictitious character of those numbers are already founded" (Fischer, 1933:113-114). In the absence of an analysis of the modal meaning and interconnections between the aspects of number and space this conclusion seems to be self-evident. Vaihinger and Fischer in particular merely used the number concept of potential infinity as a yardstick to judge the (onto-)logical status of the actual infinite.

As soon as the mutual relationships between the aspects of number and space are contemplated from the perspective of the numerical aspect, new possibilities emerge. What should first of all be kept in mind, is that the spatial order of simultaneity determines the original meaning of the whole-parts relation. The practice to speak of integers, that is of *whole* numbers, clearly imitates the "wholeness" part of the whole-parts relation. When the direction is reverted and turned from *wholeness* towards the parts of the whole-parts relation we encounter the *infinite divisibility* of a (continuous) whole and this move underlies the equally meaningful practice to speak of fractions ("broken" numbers; the rational numbers). Only the *natural numbers* appear to be envisaged without imitating some or another spatial feature. But what is the case with the potential and the actual infinite?

The most basic difference between these two kinds of infinity is that they rely upon what Dooyeweerd identified as distinct *modal time orders*, namely the *numerical time order of succession* and the *spatial time order of at once*. The lack of an intuitive understanding of the traditional phrases *potential infinity* and *actual infinity* is immediately rectified when the numerical and spatial time orders are involved. The obvious expressions then are: the *successive infinite* and the *simultaneous infinite*. In German this distinction is captured by the phrases "das sukzessiv Unendliche" and "das simultan Unendliche," while their Latin equivalents read *infinitum successivum* and *infinitum simultaneum* (see Maier, 1964:77-79). A succinct rendering of the simultaneous infinite is: the *at once infinite*.

The key element in this designation is given in the appeal to the spatial time order for this reference indicates that in the at once infinite the meaning of the numerical aspect is deepened through this anticipation to space. Pointing towards the spatial time order does not belong to the *constitutive* meaning of the arithmetical aspect. Only when its meaning is *regulatively* opened up do we encounter the *regulative hypothesis* of the at once infinite.

Whereas the number concept of successive infinity is constitutive, the number idea of the at once infinite is regulative. As regulative hypothesis this deepened structural

35 "*Aus dem Paradies, das Cantor uns geschaffen, soll Niemand uns vertreiben können*."

element within the numerical aspect enables mathematical thought to observe any successively infinite sequence of numbers as if it is given at once, as an infinite whole or totality.

This regulative hypothesis prompted both Lorenzen and Bernays to allude to the idea of a fiction. According to Paul Lorenzen the meaning of actual infinity as attached to the "all" shows the employment of a fiction – "the fiction, *as if* (my emphasis – DFMS) infinitely many numbers are given" (Lorenzen, 1952:593). However, Paul Bernays did see the essentially *hypothetical* character of the opened up meaning of number, clearly revealed in his statement:

> The position at which we have arrived in connection with the theory of the infinite may be seen as a kind of the philosophy of the 'as if'. Nevertheless, it distinguishes itself from the thus named philosophy of Vaihinger fundamentally by emphasizing the consistency and trustworthiness of this formation of ideas, where Vaihinger considered the demand for consistency as a prejudice ... (Bernays, 1976:60).

Paul Lorenzen describes the modern conception of real numbers in terms of the at once infinite in a way which strikingly reflects the spatial time order of at once: "and thus every real number as such is represented as an infinite decimal fraction as if the infinite number of digits all existed at once (*auf einmal existierten*)."[36]

If succession and simultaneity are irreducible, reflecting the irreducibility of the aspects of number and space, then the idea of an *infinite totality* cannot simply be seen as the *completion* of an infinite succession. Therefore, when Dummett refers to the classical treatment of infinite structures "as if they could be completed and then surveyed in their totality" he mistakenly equates this "infinite totality" with "the entire output of an infinite process" (Dummett, 1978:56). The idea of an infinite totality once and for all *transcends* the concept of the successive infinite.

The fact that Cantor explicitly describes the actual infinite as a constant quantity, firm and determined in all its parts (Cantor, 1962:401) underscores the implicit appeal to the meaning of space in the idea of a set. Throughout the history of Western philosophy and mathematics, all supporters of the idea of *actual infinity* (the at once infinite) implicitly or explicitly employed some form of the *spatial order of simultaneity*. What should have been used as an anticipatory regulative hypothesis (the idea of actual infinity), was often (since Augustine) reserved for God or an eternal being, accredited with the ability to oversee any infinite multiplicity all at once, also found in the thought of Bernays (as noted above).

Although modern (axiomatic) set theory (Cantor, Zermelo, Fraenkel, Hilbert, Ackermann, Von Neumann) largely pretends to be purely atomistic the structure of set theory actually implicitly (in the undefined term "set") "borrows" the *whole-parts relation* from space. This explains why Hao Wang informs us that Gödel speaks of sets as being "quasi-spatial" – and then adds the remark that he is not sure whether Gödel would have said the "same thing of numbers" (Wang, 1988:202)!

36 "Man stellt sich vielmehr die reellen Zahlen als alle auf einmal wirklich vorhanden vor – es wird sogar jede reelle Zahl als unendlicher Dezimalbruch selbst schon so vorgestellt, als ob die unendlich vielen Ziffern alle auf einmal existierten" ["Much rather one imagines that all real numbers are really present – and even every real number is represented as an infinite decimal fraction as if the infinitely many digits all existed at once"] (Lorenzen, 1972:163).

Without entering into an analysis of the intellectual journey of Edmund Husserl (1859-1938)[37] it will suffice to recall that he truly wrestled with what he experienced as the *crisis* of Europe and the disciplines. Close to the end of his life Gödel spent much time in studying the works of Husserl (see Yourgrau, 2005:170, 182). The fact that we have seen that Gödel advanced ideas concerning *primitive terms* and *coherence* which closely approximate the principles of sphere-sovereignty and sphere-universality may be linked to the fact that Dooyeweerd acknowledged that during his early development he was strongly influenced by Husserl (see Dooyeweerd, 1997-I:v).

Interestingly the so-called set theoretical antinomies made Gödel cautious for the idea of the totality character of sets, at least in the case of infinite "all" claims. We noted earlier that he remarked that the "naively" employed concept of a set "has not led to paradoxes": "This concept of set, according to which a set is anything obtainable from integers (or some other well-defined objects) by iterated application of the operation of 'set of ', and not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever" (quoted by Yourgrau, 2005:137). From this quotation it is clear that Gödel and Yourgrau do not realize that the idea of a *set* inherently contains an appeal to the whole-parts relation originally given within the spatial aspect.

Once the forward-pointing (anticipatory) hypothesis (referring number to space) is recognized as a disclosed or deepened approach, it is clear that any successively infinite sequence of numbers may be viewed *as if* they are all given at once, as an infinite whole or an infinite totality. This regulative hypothesis is not purely theoretical, in the sense of solely being an intellectual construct, for ultimately it explores and opens up ontic features of reality. In respect of the logical itself Russell equally maintains an ontic appeal: "Logic, I should maintain, must no more admit a unicorn than zoology; for logic is concerned with the *real world* (my emphasis – DS) just as truly as zoology, though with its more abstract and general features" (Russell, 1919: 169). Gödel quotes the second half of this statement with approval – see Wang (1988:313).

Yet we saw that Gödel sensed that the notion of a set echoes something of the meaning of space., for he said that sets are "quasi-spatial." In terms of Dooyeweerd's theory of modal aspects one might say that the mathematical theory of sets, proceeding from Cantor's idea of a set as a multiplicity of truly distinct elements united into a whole (*zu einem Ganzen* – Cantor, 1962:282), must be seen as a spatially deepened mathematical theory. In other words, set theory is based upon a spatial anticipation within the modal structure of the numerical aspect. This statement is equivalent to what we mentioned in quoting that Bernays stated that the idea of the continuum is a geometrical idea which is expressed in the language of arithmetic by analysis (Bernays, 1976:74). One is therefore equally justified in viewing mathematical analysis as a spatially disclosed mathematical theory. What do these considerations imply for the claims of arithmeticism?

6. Arithmeticism

In our discussion thus far the problem of arithmeticism frequently surfaced. Both Bernays and Dooyeweerd objected to its claims. Yet the striking point is that Dooyeweerd rejected the at once infinite while Bernays defended it without attaching to it the arithmeticistic consequences of those who followed Cantor in his pursuit of what he considered to be a purely arithmetical concept, that of a continuum of points

37 The dialectical devlopment of Husserl is analyzed in Strauss, 2009:625-631.

(Cantor, 1962:192). Bernays realized that Brouwer and his intuitionistic school advanced an alternative understanding of "the continuum," one in which the viewpoint of a strict arithmetization made way for an over-accentuation of the geometrical perspective (Bernays, 1976:173).

Bernays remarked that the weakest Platonist assumption within arithmetic is that of the totality of the integers [whole numbers – "ist die von der Gesamtheit der ganzen Zahlen"] (Bernays, 1976:63). Nonetheless we noted that it is precisely the totality character of continuity that withstand every arithmeticistic attempt to reduce continuity completely to discreteness – he repeatedly emphatically stated that the idea of the continuum originally is a geometrical idea.

Perhaps his most radical questioning of arithmeticism within mathematics is found in his statement: "The arithmetizing monism within mathematics is an arbitrary thesis. That the field of investigation of mathematics solely derives from representations of number is not at all shown" (Bernays, 1976:188). His subsequent remarks on this page reveals his own account of what Dooyeweerd has called the sphere-sovereignty and irreducibility of the various modal aspects of reality: "Much rather it is presumably the case that concepts such as those of a continuous curve and a surface, which are disclosed particularly in topology, cannot be reduced to representations of number."[38] In connection with the whole-parts relation Hermann Weyl points out that having broken the continuum apart in isolated points, contemporary analysis had to take recourse to the concept of an environment: "By contrast it belongs to the essence of the continuum that every part of it allows an unlimited continued division; ... To reflect the continuous coherence of points contemporary analysis, since it broke the continuum apart in a set of isolated points, took recourse to the environment concept."[39]

At this point something apparently strange surfaces. When one compares the criteria stipulated by Aristotle for continuity, it turns out that the Cantor-Dedekind description of continuity still adheres to them. Yet Aristotle considers it impossible to explain the continuity of a straight line in terms of the (infinite) number of its points. If "that which is infinite is constituted by points, these points must be either continuous or in contact with one another" (Physica, 231a29-31; Aristotle, 2001:316). Points are, however, indivisible (a point has no parts), while "that which is intermediate between two points is always a line" (Physica, 231b8; Aristotle, 2001:316). According to Aristotle it is clear that "everything continuous is divisible into divisibles that are infinitely divisible: for if it was divisible into indivisibles, we should have an indivisible in contact with an indivisble, since the extremities of things that are continuous with one another are one [i.e. the *same* – DFMS] and are in contact" (Physica, 231b15ff.; Aristotle, 2001:317).

The new arithmeticistic tendency which emerged at the beginning of the 19th century aimed at an *arithmetical definition* of *spatial continuity*. Bernard Bolzano already illuminates this tendency in § 38 of his work on the paradoxes of the infinite (postumously published in 1851). He mentions the objection that a circle would appear to be

38 "Der arithmetisierende Monismus in der Mathematik ist eine willkürliche These. Daß die mathematische Gegenständlichkeit lediglich aus der Zahlenvorstellung erwächst, ist keineswegs erwiesen. Vielmehr lassen sich vermutlich Begriffe wie diejenigen der stetigen Kurve und der Fläche, die ja inbesondere in der Topologie zur Entfaltung kommen, nicht auf die Zahlenvorstellungen zurückführen."

39 "Hingegen gehört es zum Wesen des Kontinums, daß *jedes seiner Teile sich unbegrenzt weiter teilen läß*t; ... Um die stetigen Zusammenhang der Punkte wiederzugeben, nahm die bisherigen Analysis, da sie ja das Kontinuum in eine Menge isolierter Punkt zerschlagen hatte, ihre zuflucht zu dem *Umgebungs*begriff" (Weyl, 1921:77).

hidden in the attempt to build extension out of parts which themselves are not extended, but is of the opinion that the problem disappears when it is realized that "each whole" "has numerous properties absent in the parts" (Bolzano, 1920:72). According to Dedekind's notion of a *cut* the "split" is greater than or equal to every element in the one set and smaller than or equal to all the elements of the other set (cf. e.g. Bartle, 1964:51).

Cantor himself refers to the relation which exists between his view of a *perfect set* and Dedekind's *cut theorem* (Cantor, 1962:194). Böhme realized that Cantor's definition of the continuum contains two stipulations which both meet the Aristotelian definition of a continuum, namely *coherence* and a characteristic which ensures the existence of *dividing points for infinite division* (Böhme, 1966:309). When only a Dedekind-cut is allowed at divisions, Böhme justifies his statement as follows:

> [W]hen a Cantorian continuum as such is divided in two by means of the indication of a point so that the one set contains those points which are in numerical value greater than or equal to the indicated point, while the other set contains those points of which the numerical values are smaller than or equal to the numerical value of the indicated point, both parts are again continuous. Such divisions are possible into infinity (due to the perfection of the continuum), and the parts are still coherent in the Aristotelian sense (i.e. their limit-points are the same) (Böhme, 1966:309).

This is a remarkable situation: the Cantor-Dedekind description of the continuum presupposes the use of the actual infinite (in particular by using the at once infinite set of real numbers), but nonetheless meets Aristotle's two requirements for a continuum – in spite of the fact that Aristotle indeed *rejects* the actual infinite and recognizes only the successive infinite! Did Aristotle after all use the at once infinite implicitly, or is the Cantor-Dedekind definition in the last instance not purely arithmetically in nature?

From our earlier considerations the problem is immediately solved. Because Aristotle argues from the perspective of the *spatial aspect* he merely needs the *successive infinite* (a retrocipation within space to the primitive numerical meaning of the successive infinite, making possible the inifinite divisibility of a continuum). Cantor and Dekind by contrast, approach the issue from the perspective of the deepened numerical aspect by using the idea of the at once infinite.

7. Why number is *not* "continuous"?

Since every attempt to reduce spatial continuity to numerical discreteness implicitly or explicitly has to employ the at once infinite we have to acknowledge that not even the real numbers are continuous. While the integers imitate the totality character of spatial continuity and fractions the part element of the spatial whole-parts relation, one can at most say that the real numbers imitate (analogically reflect) the nature of spatial continuity.

Bernays points at two tendencies apparently striving in opposite directions, exemplified in the homogeneity of the continuum on the one hand and the discrete determination of magnitudes on the other, for within the sequence of numbers every element is to be seen as an individual with its own particular properties. Within the context of space one merely has the succession of a repitition of what is the same. The challenge for constructing a theory of the continuum is to accomplish a reconciliation of these two opposing tendencies. In the oparational treatment the one becomes so dominant that the homogeneity of the other is not sufficiently acknowledged.[40]

Laugwitz also points out that although there are a multiplicity of lines and surfaces in space, neither points, nor lines, nor surfaces display, like number, characteristic feartures.[41] On the next page he states that the set concept was designed in such a way that what is continuous escapes from its grasp. Cantor contemplated that a set combines clearly distinct elements into a whole, from which it is clear that the *discrete* rules: "The set concept from the beginning has been designed in such a way that the continuous withdraws itself from ist grasp, for according to Cantor in the case of a set it ought to concern a 'combination' of clearly distinct things ... – the discrete governs."[42]

8. Concluding remark

It is indeed remarkble to what an extent, in spite of the obvious differences that are still present between them, the thought of Bernays, Dooyeweerd and Gödel converges, particularly in respect of the acknowledgement of the difference between the discrete and the continuous, the foundational position of number, the fact that continuity is derived from space, the inevitability to recognize what is primitive (and indefinable) and at once to account for the coherence of what is unique, as well as observing the quasi-spatial character of sets. It is clear that Dooyeweerd's theory of modal aspects provides a philosophical framework that exceeds his own restrictive understanding of infinity and at the same time makes it possible to account for key insights found in the thought of Bernays and Gödel. When Laugwitz says that discreteness rules within the sphere of the numerical, he says nothing more than what Dooyeweerd had in mind with his idea that *discrete quantity*, as the meaning-nucleus of the arithmetical aspect, qualifies every element within the structure of the quantitative aspect. And when Bernays says that analysis expresses the idea of the continuum in arithmetical language, he intends what from a Dooyeweerdian perspective is seen as the spatial anticipation within the structure of the arithmetical aspect.

The view of the at once infinite in terms of an "as if" approach (Bernays), that is, as a regulative hypothesis through which every successively infinite multiplicity of numbers could be envisaged as being giving all at once as an infinite totality, provides a sound understanding of the at once infinite and makes it plain why every form of arithmeticism fails. Such attempts have to make an appeal to Cantor's proof on the non-denumerability of the real numbers – and this proof prsupposes the use of the at once infinite. Without this assumption, which therefore pre-supposes the spatial order

40 The free translation in the text is based upon the following explanation by Bernays: "Die wiederstrebenden Momente für die zu wählende Begriffsbildung sind diejenigen der in der Idee des Kontinuums intendierten Homogenität einerseits und des Erfordernisses der begrifflichen Unterscheidungen für die Maßbestimmung der Größen anderseits. In der Zahlenreihe ist arithmetisch betrachtet jedes Element ein Individuum mit seinen ganz besonderen eigenschaften; geometrisch angesehen haben wir hier bloß die Aufeinanderfolge von sich wiederholenden Gleichartigen. Die Aufgabe bei der Bildung einer Theorie des Kontinuums ist nicht einfach ein Beschreiben, sondern eine Versöhnung zweier auseinanderstrebender Tendenzen. Bei der operativen Behandlung erhält die eine so sehr das Übergewicht, daß dabei die Homogenität zu kurz kommt" (Bernays, 1976:115).

41 "... es ist nicht möglich, einen Punkt vom andern zu unterscheiden, keine Gerade oder Ebene ist vor einer anderen Gerade oder Ebene ausgezeichnet" (Laugwitz 1986:9).

42 "Der Mengenbegriff ist von vornherein so angelegt worden, daß sich das Kontinuierliche seinem Zugriff entzieht, denn es soll sich nach Cantor bei einer Menge ja handeln um eine 'Zusammenfassung' wohlunterschiedener Dinge ... – das Diskrete herrscht" (Laugwitz, 1986:10).

of simultaneity, the real numbers collapses into *denumerability*.[43] While rejecting the actual infinite, intuitionism interprets Cantor's diagonal proof of the non-denumerability of the real numbers in a constructive (non-denumerable) sense – cf. Heyting (1971:40), Fraenkel et al. (1973:256,272), and Fraenkel (1928:239 note 1).

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43 To reach the conclusion of non-denumerability, every constructive interpretation falls short – simply because there does not exist a constructive transition from the potential to the actual infinite (cf. Wolff, 2010).

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