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The Significance of a Non-Reductionist Ontology for the Discipline of Mathematics: A Historical and Systematic Analysis

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Abstract A Christian approach to scholarship, directed by the central biblical motive of creation, fall and redemption and guided by the theoretical idea that God subjected all of creation to His Law-Word, delimiting and determining the cohering diversity we experience within reality, in principle safe-guards those in the grip of this ultimate commitment and theoretical orientation from *absolutizing* or *deifying* anything within creation. In this article my over-all approach is focused on the one-sided legacy of mathematics, starting with Pythagorean arithmeticism ("everything is number"), continuing with the geometrization of mathematics after the discovery of irrational numbers and once again, during the nineteenth century returning to an arithmeticistic position. The third option, never explored during the history of mathematics, guides our analysis: instead of reducing space to number or number to space it is argued that both the uniqueness of these two aspects and their mutual coherence ought to direct mathematics. The presence of different schools of thought is highlighted and then the argument proceeds by distinguishing numerical and spatial facts, while accounting for the strict correlation of operations on the law side of the numerical aspect and their correlated numerical subjects (numbers). Discussing the examples of $2 + 2 = 4$ and the definition of a straight line as the shortest distance between two points provide the background for a brief sketch of the third alternative proposed (inter alia against the background of an assessment of infinity and continuity and the vicious circles present in contemporary mathematical arithmeticistic claims).

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1 Overall Perspective

The attempt to reduce what is truly unique to something else results from the deification of something or some aspect within creation, normally accompanied by imperialistic "all"-claims such as, "everything is number," "everything is matter," "everything is feeling," "everything is historical" or "everything is interpretation." The distortions thus created inevitably result in insoluble *antinomies*.

A Christian approach to scholarship, directed by the central biblical idea that God is the Creator of all the laws found in the cosmos in principle safeguards theories from absolutizing or deifying anything in creation.

1.1 The Contemporary Intellectual Climate

In spite of the decline of positivism within the domain of the *philosophy of science* many scholars in the various academic disciplines (special sciences) still advocate its *neutrality postulate*. As examples of "exact sciences" mathematics and physics are the most frequently cited. These two disciplines, according to the positivistic view, are *objective* and *neutral*—they rule out the possibility of any and all presuppositions exceeding the boundaries of these "exact" disciplines. Alternatively, insofar as *historicism* and its relativistic consequences gave rise to what is known as *postmodernism*, grand stories ("meta-narratives") are questioned and truth uprooted—every person has her own "story" to tell.

Amidst all of this the leading philosophers of science during the twentieth century increasingly acknowledged the inevitability of an ultimate commitment in scholarship as well as the presence of a (philosophical) theoretical view of reality underlying all academic endeavours. Some of them explicitly reject *reductionism*. Popper straightforwardly states: "As a philosophy, reductionism is a failure" (Popper 1974, p. 269). In order to capture problematic situations within the disciplines (and their logic) the term *reductionism* emerged by the middle of the twentieth century. In 1953 Quine used it in his discussion of "The Verification Theory and Reductionism" (see Quine 1953, 37ff) and in the early seventies the work "Beyond Reductionism" appeared (see Koestler and Smythies 1972). Smith (1994) considers the scientist-philosopher Michael Polanyi to be "perhaps the severest and most comprehensive critic of reductionism" because he "was a major scientist of this century and was drawn into philosophical debate primarily because of the threat to scientific freedom, political democracy, and to humane values that he saw in reductionism". To this he adds the remark:

> His works *The Contempt of Freedom*, *The Logic of Liberty*, *Science Faith and Society*, *Personal Knowledge*, and *The Tacit Dimension* have as a common

> theme the criticism of reductionism in all its scientific, cultural and moral forms.

The best way to challenge both positivism (objectivity and neutrality) and postmodernism (historicism and relativism) is to confront the supposedly "exact" sciences, mathematics and physics, with the implications of a non-reductionist Christian philosophy.

2 Mathematics

> Practicing mathematicians, consciously or not, subscribe to some philosophy of mathematics (if unstudied, it is usually inconsistent) (Monk 1970, p. 707)

2.1 Are There Different Standpoints in Mathematics?

Before we investigate relevant historical perspectives and systematic distinctions it is worth challenging the following claim of Fern:

> Mathematical calculations are paradigmatic instances of universally accessible, rationally compelling argument. Anyone who fails to see "two plus two equals four" denies the Pythagorean Theorem, or dismisses as nonsense the esoterics of infinitesimal calculus forfeits the crown of rationality (Fern 2002, pp. 96–97).

We do this by quoting a number of statements.

The mathematician Kline writes:

> The developments in the foundations of mathematics since 1900 are bewildering, and the present state of mathematics is anomalous and deployrable. The light of truth no longer illuminates the road to follow. In place of the unique, universally admired and universally accepted body of mathematics whose proofs, though sometimes requiring emendation, were regarded as the acme of sound reasoning, we now have conflicting approaches to mathematics (Kline 1980, pp. 275–276)

In respect of formalization in intuitionistic mathematics the Dutch logician Beth remarks:

> Meanwhile, for the intuitionists this formalization has in no way the meaning of a foundation as it does for the logicists. On the contrary, formalistic expression is in a position to produce no more than an inadequate picture of intuitionism (Beth 1965, p. 90).

The intuitionistic mathematician, Heyting, explains what is basic to intuitionism

> every logical theorem … is but a mathematical theorem of extreme generality; that is to say, logic is a part of mathematics, and can by no means serve as a foundation for it (Heyting 1971, p. 6).

Of course intuitionism represents an authentic mathematical stance:

> The intuitionists have created a whole new mathematics, including a theory of the continuum and a set theory. This mathematics employs concepts and makes distinctions not found in the classical mathematics (Kleene 1952, p. 52).

In fact intuitionism created an entirely *new mathematics*. Beth explains:

> It is clear that intuitionistic mathematics is not merely that part of classical mathematics which would remain if one removed certain methods not acceptable to the intuitionists. On the contrary, intuitionistic mathematics replaces those methods by other ones that lead to results which find no counterpart in classical mathematics (Beth 1965, p. 89).

Perhaps the most perplexing observation comes from Stegmüller:

> The special character of intuitionistic mathematics is expressed in a series of theorems that contradict the classical results. For instance, while in classical mathematics only a small part of the real functions are uniformly continuous, in intuitionistic mathematics the principle holds that any function that is definable at all is uniformly continuous" (Stegmüller 1970, p. 331).

2.2 Two Apparently Simple Questions With 'Self-Evident' Answers

(i) Is $2 + 2 = 4$? and (ii) is a straight line the shortest distance between two points?

We may relate question (i) to the idea that mathematics is objective and neutral—as asserted by Fern in the quotation given above.

The statement that "a straight is line the shortest distance between two points" indeed seems to be as self-evident as the statement that "$2 + 2 = 4$". In an earlier phase of his development Bertrand Russell 'corrected' this definition: "A straight line, then, is not the shortest distance, but is simply the distance between two points" (Russell 1897, p. 18). The three key terms in this statement concern *spatial* configurations, namely the terms 'line', 'point' and 'shortest'. Yet the crucial element maintained in Russell's improved definition echoes something of our awareness of numerical relations: *distance*.[1] If this is indeed the case it may turn out that an analysis of this statement will at once get entangled in the consideration of arithmetical and spatial issues, which means that it cannot be analyzed purely in *spatial* (or *geometrical*) terms.

[1] The focus of our considerations will be on the interconnections between space and number. We shall argue that there are structural features that are inherent in these two facets of reality prior to the actual definition of *metrical* spaces (in 1906 by Fréchet). Mac Lane accepts space as "something extended" and on the basis of the notion of "distance" defines a *metric space* (see Mac Lane 1986, pp. 16–17). It is clear that the notions of *extension* and *distance* **precede** the definition of a metrical space. An explanation of the mutual relation between discreteness and continuity within a *topological* context requires a different argument. A starting-point for such a discussion is found in White (1988, pp. 1–12).

2.3 Historical Detour

Early Greek mathematics followed the arithmeticistic approach of the Pythagorean school with its claim that "everything is number." Although the Pythagoreans believed that numerical relationships ordered the cosmos, they discovered that geometrical figures and lines can be construed that cannot be expressed by the relation between integers. The discovery of incommensurability by Hippasus of Metapont (450 BC) therefore caused a crisis since within the assumed form-giving function of number the formless (infinite) was revealed. Laugwitz remarks: "Every numerical relationship allows for a geometric representation, but not every line-relationship can be expressed numerically. This established the primacy of geometry over arithmetic and as a result the Books of Euclid treat the theory of numbers as a part of geometry" (Laugwitz 1986, p. 9).[2] This geometrization of mathematics inspired a space metaphysics lasting at least until Descartes and Kant. During the nineteenth century, however, Cauchy, Weierstrass, Dedekind and Cantor once again pursued the path of an *arithmeticistic* approach. Of particular significance in this regard is set theory as it was developed by Cantor (including his theory of *transfinite arithmetic*).

When Russell and Zermelo independently discovered the fundamental inconsistency of Cantor's set theory in 1900 and 1901, mathematics gave birth to three schools of thought, namely the *logicist* school (Russell, Gödel), the *intuitionist* school (Poincaré, Brouwer, Heyting, Weyl and Dummett), as well as the *axiomatic formalist* school (guided by the foremost mathematician of the twentieth century, David Hilbert and still largely dominating the scene of contemporary mathematics).

The first edition of Kant's influential work, The Critique of Pure Reason, appeared in 1781. What is remarkable is that the main systematic subdivisions of this work provide the springboard for the three diverging trends in 20th century mathematics just mentioned—intuitionism (exploring the "transcendental aesthetic" of the CPR), logicism (oriented to the "transcendental analytic") and axiomatic formalism (affirming the thrust of the "transcendental dialectic")—as explicitly acknowledged by each of their main representatives.[3] We already noted that the fundamental differences within the discipline of mathematics caused a situation where what is true within intuitionistic mathematics may be false within formalism, while what is mathematically accepted by formalism, such as Cantor's theory of transfinite numbers, is rejected as a phantasm by intuitionism (see Heyting 1949, p. 4) and as non-existent.[4]

[2] In connection with the history of the concept of matter we shall return to Greek philosophy.

[3] Already Kronecker, a prominent early intuitionist (and contemporary of Cantor) defended a Kantian view on the a priori nature of arithmetic while denying it in respect of geometry and mechanics (see Kronecker 1887, p. 265). Al great master of twentieth century intuitionist mathematics, Brouwer also holds: "However weak the position of intuitionism seemed to be … it has recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time" (Brouwer 1964, p. 69). Gödel's remarks that there is a close relationship between the concept of a set explained in footnote 14 and the categories of pure understanding in Kant's sense" (see Gödel 1964, p. 272). Finally, Hilbert refers to Kant's understanding of *reason ideas* in order to justify his own justification of the mathematical usefulness of the actual infinite (see Hilbert 1925, p. 190.

[4] Just compare the remarks quoted above concerning different standpoints in mathematics.

In 1900 the French mathematician, Poincaré, made the proud claim that mathematics had reached absolute rigour. In a standard work on the foundations of set theory, however, we read: "ironically enough, at the very same time that Poincaré made his proud claim, it has already turned out that the theory of the infinite systems of integers—nothing else but part of set theory—was very far from having obtained absolute security of foundations. More than the mere appearance of antinomies in the basis of set theory, and thereby of analysis, it is the fact that the various attempts to overcome these antinomies, …, revealed a far-going and surprising divergence of opinions and conceptions on the most fundamental mathematical notions, such as set and number themselves, which induces us to speak of the third foundational crisis that mathematics is still undergoing" (Fraenkel et al. 1973, p. 14).

The history of Gottlieb Frege's work is perhaps the most striking. In 1884 he published a work on the foundations of arithmetic. After his first Volume on the basic laws of arithmetic appeared in 1893, Russell's discovery (in 1900) of the antinomous character of Cantor's set theory for some time delayed the publication of the second Volume in 1903—where he had to concede in the first sentence of the appendix that one of the corner stones of his approach had been shaken. Russell considered the set C with sets as elements, namely all those sets A that do not contain themselves as an element. It turned out that if C is an element of itself it must conform to the condition for being an element, which stipulates that it cannot be an element of itself. Conversely, if C is not an element of itself, it obeys the condition for being an element of itself.

Close to the end of his life, in 1924/25, Frege not only reverted to a geometrical source for mathematical knowledge, but also explicitly rejected his initial logicist position. In a sense he completed the circle—analogous to what happened in Greek mathematics after the discovery of irrational numbers. In the case of Greek mathematics this discovery prompted the geometrization of their mathematics, and in the case of Frege the discovery of the untenability of his "Grundlagen" also inspired him to hold that mathematics as a whole actually is geometry:

> So an a priori mode of cognition must be involved here. But this cognition does not have to flow from purely logical principles, as I originally assumed. There is the further possibility that it has a geometrical source…. The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis—a geometrical one in fact—so that mathematics in its entirety is really geometry (Frege 1979, p. 277).

What is therefore the upshot of the history of mathematics? It emerged under the spell of Pythagorean arithmeticism ("everything is number"), then, owing to the discovery of irrational numbers (incommensurability) it experienced a fundamental geometrization and during the nineteenth century it once again explored the avenue of arithmeticism, thus closing the circle of arithmeticism. Finally, in the thought of Frege, there was a return to a spatial basis for mathematics.

2.4 Starting-Points for a Third Alternative?

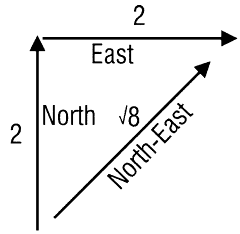
From the perspective of a non-reductionist ontology there is an obvious alternative never pursued throughout the history of mathematics:

> Accept the uniqueness and irreducibility of number and space as well as their mutual interconnectedness and coherence.

In order to highlight this alternative way we may use the above-mentioned argument regarding $2 + 2 = 4$ as our angle of approach. This will introduce considerations stemming both from the domains of number and space, particularly through the introduction of another ‘sum’—one in which we may suggest that “$2 + 2$” is not equal to 4 but equals $\sqrt{8}$.

2.5 Numerical and Spatial Addition in the Context of the Law-Subject Distinction

The conviction that mathematics is objective and neutral could be defended by pointing out that $2 + 2 = 4$ can be confirmed by counting on our fingers. Apparently this specified addition conclusively confirms the soundness of the initial statement that $2 + 2$ is equal to 4. Unfortunately the issue is more complicated than it may seem at first sight, for the alternative assertion, namely that $2 + 2 = \sqrt{8}$ implicitly changed the *context of addition*. When a person walks two miles north and afterwards 2 miles east, then that person will be $\sqrt{8}$ miles away from the initial point of departure. This context concerns an instance of *spatial addition* that is mathematically treated in *vector analysis*, where a vector possesses both *distance* (magnitude) and *direction*. One can capture this altered context by underscoring the numerals involved in order to specify the fact that we are dealing with vectors: $\underline{2} + \underline{2} = \sqrt{8}$. The upshot is that we now clearly have two different *kinds of facts* related to addition at hand: a *numerical* fact (designated as $2 + 2 = 4$) and a *geometrical* fact (in the “right-angle-case” designated as $\underline{2} + \underline{2} = \sqrt{8}$). In order to capture the specifications of this example one may construct the following figure:



These facts are not *unqualified*—that is to say, they are *distinct* because they are differently qualified or structured, respectively as *numerical* and as *spatial* facts. They are therefore not simply ‘facts’ in themselves. In their factuality they are *delimited* by alternative order-determinations. The operation of numerical addition displays an order-determination different from the operation of spatial addition, as is clearly manifested in the alternative sums: 4 and $\sqrt{8}$. In our example the underlying "order diversity" therefore makes possible the indicated distinction between numerical and spatial facts.

But there is something else present in this distinction between these two kinds of facts, namely the reference to the *operation of addition*. Modern mathematical set theory normally first of all approaches this domain in terms of the algebraic structure of *fields*—where the (binary) operations called addition (+) and multiplication (×) meet the field axioms (specified as *laws*).[5]

Let us take one step back and initially extract from this mathematical practice merely the operations (laws) of addition and multiplication. The fact that addition and multiplication within a specific system of numbers (such as the *system* of natural numbers) yield numbers still belonging to the initial set, can also be mathematically articulated by saying that the *system of numbers* under consideration is *closed* under the operations (laws) of *addition* and *multiplication*. Since ancient Greek philosophy it was understood that conditions (laws) and whatever meets these conditions are both distinct and strictly correlated. The most basic instance of such a strict correlation between (arithmetical) laws and arithmetical subjects (numbers) conditioned by these laws is found in the system of natural numbers. It is immediately evident that the addition and multiplication of any two natural numbers once more yield natural numbers (s = system; t = set):

system of natural numbers N_s	operations/laws:	$(+, \times)$
	numerical subjects:	$N_t = (1, 2, 3, \ldots)$

The designation 'system' therefore comprises both *arithmetical laws* and *arithmetical subjects*—in the sense that the laws (operations) not only *determine* the behavior of the subjects but also *delimit* them. What has been explained above therefore means that the system of natural numbers finds its *determination* and *delimitation* in the operations of addition and multiplication that are closed over the set of natural numbers—in the sense that adding or multiplying any two natural numbers will always yield another natural number. The ultimate presupposition of these operations is found in the numerical order of succession. The Peano axioms (for the positive integers) yield a mathematical articulation of this primitive arithmetical order of succession. The correlation of the operations of addition and multiplication and their delimiting and determining role in respect of numerical subjects are consistent with Peano's axioms because they are entailed in the *complete ordered field of real numbers* (see Berberian 1994, p. 230).

Introducing further arithmetical laws or operations will invariably call for additional (correlated) numbers that are factually subjected to the determining and delimiting arithmetical laws. For example, if the operation of subtraction is added to those of addition and multiplication, the correlating set of integers (Z_t) is

[5] A *field* is defined as a set F such that for every pair of elements a, b the sum $a + b$ and the product ab are still elements of F subject to the associative and commutative laws for addition and multiplication, and combined to the presence of a *zero element* and a *unit* (or *identity*) *element* (see Bartle 1964, p. 28; Berberian 1994, 1ff). This definition of a field is then expanded to that of an ordered field and it is finally connected to the idea of completeness. The existence of least upper bounds differentiate the real numbers from all other ordered fields" (Berberian 1994, pp. 11–12).

constituted—and considered in their correlation this yields the *system of integers* (see Ebbinghaus et al. 1995, p. 19).

system of integers I_s	operations/laws:	$(+, \times, -)$
	numerical subjects:	$I_t = (0, +1, -1, +2, -2, \ldots)$

Likewise, extending the arithmetical operations by introducing *division* the correlating *set* of fractions is needed within the *system* of rational numbers.[6]

system of Q_s	operations/laws:	$(+, \times, -, \div -$ and the condition that $b \neq 0)$
rational numbers	numerical subjects:	$Q_t = (a/b; a, b \in Z_t / b \neq 0)$

Against this background it is clear that the systematic arithmetical statement $2 + 2 = 4$ does not designate a "brute fact" (a fact "in itself," "an sich"), since the factual relation specified for numerical subjects (selected from the set of natural numbers) that are involved in it, exhibits the *measure* (orderliness) of the numerical law of addition. One can also say that this statement conforms to the determining and delimiting effect of the arithmetical law of addition. Consequently, the statement that $2 + 2$ is equal to 4 concerns a law-conformative (arithmetical) state of affairs—it displays a specific lawfulness or orderliness for it meets the conditions set by the presupposed arithmetical order.

If there are multiple laws known to be *arithmetical laws* then one may speak of a unique *sphere of arithmetical laws* strictly correlated with diverse arithmetical subjects (sets of numbers) subjected to these laws. Another way to capture this situation is to speak of a numerical sphere in which arithmetical laws are strictly correlated with arithmetical subjects (numbers); in other words within this numerical domain a distinction is made between its *law-side* (order side) and its *factual side*. Myhill, who appreciates Brouwer as the originator of "constructive mathematics," introduces the notion of a 'rule' (the equivalent of what we have designated as "law-side") as "a primitive one in constructive mathematics"; "We therefore take the notion of a rule as an undefined one" (Myhill 1972, p. 748). (Myhill received his Harvard Ph.D. under W.V.O. Quine). In his encompassing introduction to set theory (the third edition), Adolf Fraenkel refers to the peculiar *constructive* definition of a set which accepts as a foundation the *concept of law* and the *concept of natural number* as *intuitively given* (Fraenkel 1928, p. 237).

It should be noted that on the view advocated here numbers and spatial figures are neither Platonic entities nor nominalist fictions, but universal (functional/aspectual) ontic conditions for whatever we experience as quantitative and as spatial. Although concrete things may have typical functions with the aspects of number and space the modal universality of each aspect exceeds concrete things.

The geometrical sum $\underline{2} + \underline{2} = \sqrt{8}$ belongs to a different domain, to a different sphere of laws, one where it is also possible to distinguish between a law-side (order

[6] This explanation, in terms of the strict correlation between operations at the law-side and numerical subjects at the factual side, is *formally* similar to the way in which Klein introduces negative numbers and fractions (by means of the reverse operations of addition and multiplication—see Klein 1932, 23ff & 29ff). Ebbinghaus et al. points out that in a paper on "Pure Number Theory" (*Reine Zahlenlehre*) Bolzano already developed a theory of rational numbers, "and in fact a theory of those sets of numbers that are *closed with respect to the four elementary arithmetic operations*" (Ebbinghaus et al. 1995, p. 22).

side) and a factual side. The sphere of spatial laws differs from the sphere of numerical laws—in an exemplary way expressed in the difference between $2 + 2 = 4$ and $\underline{2} + \underline{2} = \sqrt{8}$.

Remark At this stage it should be mentioned that the aim of our analyses is not in the first place directed at interconnections between different mathematical sub-disciplines. The goal is to show that number and space are not only unique and irreducible aspects of (ontic) reality, but also to argue that they mutually cohere in many ways (eventually highlighted with reference to what will be designated as *analogies* on the law-side and on the factual side of these aspects). Whenever interconnections between mathematical sub-disciplines are highlighted the aim is to demonstrate the underlying ontic interconnections between the aspects of number and space.

2.6 Distance: Highlighting The Mutual Coherence Between Number and Space

We may now return to the mentioned key element in the modified definition given by Russell, *distance*: a line "is simply the *distance* between two points." The after-effect of the Greek geometrization of mathematics is seen in the long-standing and persistent use of the term 'Größe' ('magnitude'—for numbers) up to nineteenth century mathematicians—such as Bolzano and Cantor (in spite of their 'arithmetizing' intentions their designation of numbers still used the gateway of the spatial aspect). Greek mathematics already indirectly wrestled with *spatial magnitudes*—such as lengths, surfaces and volumes—although the *ratios* contemplated by them were treated in non-numerical contexts. By comparing spatial figures (such as line-segments, surfaces and solids) Euclidean geometry used ratios of magnitudes within a non-numerical context (sometimes a physical one) in their measurements. Naturally the Greeks were fully aware of specific *numerical properties* of spatial figures, because otherwise they would not have had a concept of a *triangle*, i.e. of a figure with *three* sides and *three* angles.

In itself this already shows that spatial figures (such as triangles) reveal an unbreakable coherence with the meaning[7] of number. Of course it should be remembered that the overemphasis of number as a mode of explanation caused the Pythagoreans to see spatial figures *as numbers*. Kurt von Fritz remarks: "Likewise, so they said, 'are' the geometrical figures in reality the numbers or bundles of numbers that constitute the length relationships of their sides; through them their form is determined and through them they can therefore be expressed" (Von Fritz 1965, p. 287). It was only through the analytical geometry of Descartes and Fermat that numerical magnitudes were eventually contemplated—assigned to line segments, surfaces and solids. Savage & Ehrlich remarks: "Euclidean geometry

[7] In the course of our analysis it will become clear that the word "meaning" fulfils a specific systematic role. It brings to expression that the meaning of an aspect only reveals itself in its coherence with other aspects. Therefore a reference to the *meaning of an aspect* intends to capture this idea of uniqueness in coherence.

compares lengths, areas, and regions by comparing physical, non-numerical *ratios* of these magnitudes and in effect uses such ratios in the place of our numbers" (see Savage and Ehrlich 1992, 1ff). However, the lack of understanding of the interconnections between number and space caused the mistaken identification of a line *with* its length (distance).

The first observation to be made in this connection is to establish that the notion of a 'line' as the 'distance' between two 'points' concerns *spatial* givens. A line is a spatial subject (configuration), not an arithmetical one. Yet the crucial question is: how can one designate the 'distance' between two points? The answer is: by specifying a *number* (for example by saying it is 3 inches long). The problem with this answer is that something *spatial*, namely a 'line', is now apparently equated with something *numerical*, namely 'distance'! In passing we note that the term 'distance' in yet a different way evinces an intrinsic connection with the meaning of number because a line is supposed to be the 'distance' between *two* points. Multiplicity ('two') is numerical; yet a multiplicity of *points* is *spatial*. Furthermore, the term 'inch' here has the function of the *unit of measurement*, i.e. the unit length. Therefore this unit is on a par with the notion of distance, because the number 1 and the number three respectively represent these two lengths. Does this mean that the domains of space and number are coinciding? If it is the case, then a question of priority arises: is space numerical (then a 'line' is identical to 'distance', i.e. to number), or is number after all spatial in nature (then number, i.e. 'distance' is identical to space, i.e. to a 'line')? As we noted this concise dilemma reflects the basic contours of the history of mathematics as a discipline. After the initial Pythagorean claim that everything is number the discovery of irrational numbers turned mathematics into geometry. Then, during the nineteenth century arithmeticism once more gained the upper hand, although Frege close to the end of his life, reverted once more to the view that mathematics essentially is geometry. The situation is further complicated by the fact that the number specified (such as '3') does not stand on its own, i.e., it appears within a non-numerical context—one in which the general issue of *magnitude* prevails, with *length* as a 1-dimensional magnitude. And to add insult to injury, we would now suddenly have to account for another spatial notion: *dimensionality*! New problematic questions are now generated, for in our example of "3 inches"—related to the extension of a line—the reference to length brought with it the (spatial) perspective of *one* dimension (length specifies magnitude in the sense of one dimensional extension). On the one hand this points at *extension* which, presumably, essentially belongs to our awareness of space, while at the same time, just as in the case of the term 'distance', it reveals a connection with number as well, for one can speak about 1-dimensional extension (magnitude; i.e. of length), 2-dimensional extension (magnitude; i.e. of area), 3-dimensional extension (magnitude; i.e. of volume), and so on. Even if priority is given to the spatial context by admitting that the distinction between different dimensions is indeed something spatial, no one can deny that in some way or other number here plays a *foundational role*, for without number the given specification (regarding 1, 2, or 3 dimensions) is unthinkable.

Clearly, the term 'distance' is embedded within the domain of space and it also evinces a strict correlation between an *order of extension* (the law-side of this domain—i.e. *dimensionality*) and *factually extended spatial subjects*—spatial figures (such as 1-dimensional ones, i.e. lines), 2-dimensional ones, i.e. areas) and 3-dimensional ones (i.e. volumes).

The complexities generated by considering an order of extension correlated with factually extended spatial subjects (spatial figures) add weight to the suggestion that although something like a line has a spatial nature, its extension reveals that its spatial meaning truly depends upon depends upon the meaning of number. The reason for this acknowledgement is found in the intrinsic role of numerical concepts that are 'coloured' by space, such as *distance* and *dimension*. Within a numerical context, such as what is known in mathematics as real analysis, one can easily dispense with the concept of distance. But textbooks on real analysis sometimes still acknowledge that the geometric meaning of the term 'distance' may be useful, for "instead of saying that $|a - b|$ is 'small' we have the option of saying that a is 'near' b; instead of saying that '$|a - b|$ becomes arbitrarily small' we can say that 'a approaches b', etc." (Berberian 1994, p. 31).

2.7 Back to Space

Two years after Russell gave his aforementioned modified definition of a line as the *distance between two points*, the German mathematician, David Hilbert, published his axiomatic foundation of geometry: *Grundlagen der Geometrie* (1899). In this work Hilbert abstracts from the contents of his axioms, based upon three *undefined* terms: "point," "lies on," and "line." Suddenly the term 'distance' disappeared. The next year, when Hilbert attended the second international mathematical conference in Paris, he presented his famous 23 mathematical problems that co-directed the development of mathematics during the twentieth century in a significant way—and in problem four he provides a formulation that opens up a new perspective on this issue, for instead of speaking of the *distance* between two points he talks of a straight line as the (shortest) *connection* of two points.[8] This choice of words completely avoids the traditional view, even found in the work of a contemporary mathematician like Mac Lane who still believes that the "straight line is the shortest distance between two points" (Mac Lane 1986, p. 17).

Hilbert's German term 'Verbindung' ('connection') does not define a line since it presupposes the meaning of continuous extension. Every part of a continuous line coheres with every adjacent part in the sense of being *connected* to it. Although it is tautological to say that the parts of a continuous line are fitted into a gapless coherence, it says nothing more than to affirm that the parts are *connected*. In this sense the connection of two distinct spatial points also highlights the presence of (continuous) spatial extension *between* the points that are connected to each other. In other words, Hilbert's formulation suggests that two points cannot be *connected* by a third point, but only by means of a line, i.e. through *continuous spatial extension*.

[8] "[Das] Problem von der Geraden als kürzester Verbindung zweier Punkte" (see Hilbert 1970, p. 302).

Combined with the primitive terms employed in his axiomatic foundation of geometry (‘line’, ‘lies on’ and ‘point’) the term ‘connection’ no longer equates a line with its distance. Once ‘liberated’ from this problematic bondage, alternative options emerge in order to account for the meaning of the term ‘distance’. If *distance* is the 1-dimensional *measure* of factual (continuous spatial) extension, then one can do two things at once:

(i) acknowledge the *spatial* context of this measure (1-dimensional magnitude) and
(ii) account for the reference to number that is evident both in the ‘1’ of 1-dimensional extension and in the (numerically specified) *length* evident in ‘distance’ as a specified (factual) spatial magnitude.

The core meaning of space, related to the awareness of extension and dimensionality, now acquires a new appreciation, further supported by the undefined nature of the term ‘line’ in Hilbert’s 1899 work. The message is clear: if the core meaning of space (extension) is indefinable and primitive, then it is impossible to attempt to *define* a line by using a term revealing a reference to what is not *original* within space, namely the number (!) employed in the specification of the ‘distance’ between two points. On the one hand, distance as the *measure of extension* of a (straight) line depends upon and presupposes the existence of the line in its primitive 1-dimensional extension and can therefore never serve as a definition of it, and on the other hand it reveals a connection with the meaning of *number*. Therefore the ‘definition’ of a (straight) line as “the distance between two points” (Russell, Mac Lane) presupposes what it wants to define and consequently begs the question.

2.8 What is Presupposed in Space?

In our discussion of the question whether or not the domains of space and number coincide, we have started by analyzing some consequences of the option that they do coincide. Aristotle already explored this possibility, but without success, because he employed the biological method of concept formation (of a *genus proximum* and *differentia specifica*) in a context where it does not fit. As *genus* his category of “quantity” is then differentiated into a *discrete quantity* and a *continuous quantity*: “Quantity is either discrete, or continuous” (*Categoriae*, 4 b 20). “Number, … is a discrete quantity” (*Categoriae*, 4 b 31). The parts of a discrete quantity have no common limit, while it is possible in the case of a line (as a continuous quantity) to find a common limit to its parts time and again (*Categoriae*, 4b 25ff, 5a 1ff). In this account the aspects of number and space are brought under one umbrella and this approach precludes an insight into the uniqueness and irreducibility of number and space.

Rejecting Aristotle’s approach calls for an acknowledgement of the fact that every specification of spatial configurations is unavoidably connected with terms reflecting in some or other way the coherence of space with the meaning of number (magnitudes and the number of dimensions). This outcome opens the way to the alternative option: of investigating the consequences of the assumption that although space and number are unique and distinct they still unbreakably cohere. The new question to be analyzed is then:

2.9 What is the Interrelation Between Space and Number?

If the measure of the factual (one dimensional) extension of a straight line could be specified by its distance, then the distance of a line not only presupposes its spatial extension since it also presupposes the intrinsic interconnection between space and number. Various mathematicians had an appreciation of this state of affairs.

But let us consider further options. The mere possibility of juxtaposing two distinct 'facts', such as the statements that $2 + 2 = 4$ and $\underline{2} + \underline{2} = \sqrt{8}$, points in the direction of acknowledging two unique domains—each with its own sphere of laws and correlated subjects. Laugwitz refers to the approach of Bourbaki according to which there is a difference between what is discrete (algebraic structures) and what is continuous (topological structures).[9]

Of course modern mathematicians are inclined to give preference to the meaning of number and infinity. Tait remarks: "Surely the most important philosophical problem of Frege's time and ours, and one certainly connected with the investigation of the concept of number, is the clarification of the infinite, initiated by Bolzano and Cantor and seriously misunderstood by Frege" (Tait 2005, p. 213). What therefore needs to be clarified is summarized in the following two issues:

(i) which one of these two domains ('poles') is more fundamental, in the sense of *foundational*, to the other?

and

(ii) how should one account for the interconnections (interrelations) between these two domains ('poles')?

2.9.1 Which Region Is More Basic?

Let us start with the approach of Bernays where he considers the way in which one can distinguish between our arithmetical and geometrical intuition. He rejects the widespread view that this distinction concerns time and space, for according to him the proper distinction needed is that between the discrete and the continuous.[10] Rucker also states: "The discrete and continuous represent fundamentally different aspects of the mathematical universe" (Rucker 1982, p. 243). Fraenkel et al. even consider the relation between discreteness[11] and continuity to be the central problem of the foundation of mathematics: "Bridging the gap between the domains of discreteness and of continuity, or between arithmetic and geometry, is a central, presumably even the central problem of the foundation of mathematics" (Fraenkel

[9] "… der Unterschied zwischen Diskretem (algebraische Strukturen) und Kontinuierlichem (topologische Strukturen)" (Laugwitz 1986, p. 12).

[10] "Es empfiehlt sich, die Unterscheidung von "arithmetischer" und "geometrischer" Anschauung nicht nach den Momenten des Räumlichen und Zeitlichen, sondern im Hinblick auf den Unterschied des Diskreten und Kontinuierlichen vorzunehmen" (Bernays 1976, p. 81).

[11] Below we shall 'liberate' the idea of 'discreteness' from the arithmeticistic habit to distinguish between discrete, dense and continuous sets—in order to allow *discreteness* to play its role as meaning-kernel of the numerical aspect that qualifies all kinds of number, even when such kinds of number may imitate spatial features (such as wholeness, divisibility and continuity).

et al. 1973, p. 211). But then the question recurs: what is the relationship between the 'discrete' and 'continuous'?[12] In terms of the distinction between the domain of number and that of space the term "pattern" in the first place derives its meaning from *spatial configurations* or *patterns*. Only afterwards can one stretch this term—metaphorically or otherwise—in order to account for quantitative relations as well.

Whatever the case may be, speaking of "discrete patterns" just as little bridges the gap between discreteness and continuity than does referring to the domain of number (where the term "domain" is also derived from the meaning of space).

Fraenkel et al. even speak of a 'gap' in this regard and add that it has remained an "eternal spot of resistance and at the same time of overwhelming scientific importance in mathematics, philosophy, and even physics" (Fraenkel et.al. 1973, p. 213). These authors furthermore point out that it is not obvious which one of these two regions—"so heterogeneous in their structures and in the appropriate methods of exploring"—should be taken as starting-point. Whereas the "discrete admits an easier access to logical analysis" (explaining according to them why "the tendency of arithmetization, already underlying Zenon's paradoxes may be perceived in [the] axiomatics of set theory"), the converse direction is also conceivable, "for intuition seems to comprehend the continuum at once," and "mainly for this reason Greek mathematics and philosophy were inclined to consider continuity to be the simpler concept" (Fraenkel et al. 1973, p. 213).

Of course the modern tendency towards an arithmetized approach (particularly since the beginning of the nineteenth century) chose the alternative option by contemplating the primary role of number. Although Frege—as mentioned above—by the end of his life equated mathematics with geometry (consistent with the just mentioned position of Greek mathematics), his initial inclination certainly was to opt for the foundational position of number. As early as in 1884 he asked if it is not the case that the basis of arithmetic is deeper than all our experiential knowledge and even deeper than that of geometry?[13]

From our discussion of the difference between an arithmetical and a spatial sum and in particular from our remarks about the term 'distance' it is possible to derive an alternative view on the order relation between the regions of discreteness and of continuity. Suppose we consider the idea that *discreteness* constitutes the core meaning of the domain of number[14] and that *continuous extension* highlights the

[12] We noted above that the problem concerning which one is more basic—number or space—cannot be solved by specifying a *genus proximum*—albeit that of Aristotle with his distinction between a discrete quantity and a continuous quantity or that of the structuralist Resnik with his distinction between *discrete patterns* and *continuous patterns* (cf. Aristotle: "Quantity is either discrete, or continuous"—*Categ*. 4 b 20; and Resnik 1997, 201ff, 224ff).

[13] "Liegt nicht der Grund der Arithmetik tiefer als der alles Erfahrungswissens, tiefer selbst als der der Geometrie?" (Frege 1884, p. 44).

[14] The arithmeticistic practice to distinguish between discrete, dense and continuous actually merely highlights disclosed structural moments through which different types of numbers, in anticipatory way, point at structural features of the spatial aspect (such as wholeness, infinite divisibility and continuity) without ever "escaping" from the qualifying role of discrete quantity. Even every real number remains distinct and unique. As Laugwitz states it: "From the outset the set concept is constructed in such a way that what is continuous escapes from its grip, for according to Cantor a set concerns the 'bringing-together' of clearly distinct things … the discrete rules" (Laugwitz 1986, p. 10). We shall return to this issue below.

core meaning of space. Then these core meanings guarantee the distinctness or uniqueness of each domain. The domain of number, with its sphere of arithmetical laws and numerical subjects, is then seen as being stamped, characterized or qualified by this core meaning of *discreteness*. Likewise the domain of space, with its sphere of spatial laws and spatial subjects, is then viewed as being qualified by the core meaning of *continuous extension*.

But we have seen that a basic spatial subject, such as a (straight) line, cannot be understood without some or other reference to the meaning of number, for observing the *measure* of the line's extension requires the notion of 'distance' that involves number. Furthermore, since a line a spatial figure is extended in 1-dimension, it clearly only has a determinate meaning in subjection to the first order of spatial extension (namely *one* dimension). We have argued that in both domains (number and space) there is a strict correlation between the law-side and the factual side. In the case of space it is therefore possible to discern a reference to number both at the *law-side* and the *factual side*. Speaking of *one* or *more* than one dimensions presupposes the meaning of number on the law-side and this mode of speech at once specifies the meaning of the one dimensional extension, i.e. *magnitude*, of something like a line where the meaning of the number employed in the designation of the *length* of the line presupposes the original (primitive) meaning of number. The domain of number therefore appears to be more basic because an analysis of the meaning of space invariably calls upon foundational arithmetical features.

This conclusion is further supported by the approach of Maddy where she argues that most recent textbooks "view of set theory as a foundation of mathematics" (Maddy 1997, p. 22; see also Felgner 1979, p. 3) and that a set theoretic foundation can "isolate the mathematically relevant features of a mathematical object" in order to find a "set theoretic surrogate" for those features (Maddy 1997, pp. 27, 34).[15] Bernays categorically asserts that "the representation of number is more elementary than geometrical representations" (Bernays 1976, p. 69).[16] In general one may view the arithmeticism of Weierstrass, Dedekind and Cantor as an (over-estimated) acknowledgement of the foundational position of the domain of number.

We may summarize the thrust of our preceding argument in favour of the foundational position of number in respect of space as follows:

The core meaning of space—namely *continuous extension*—entails *factual* extension in one or more *dimensions*; and specifying "one or more" dimensions presupposes the natural numbers 1, 2, 3, … At the same time the 1-dimensional extension of a straight line comes to expression in the *measure* of this extension, designated as its *length*—and the latter (its *length*) is specified by using a *number*—

[15] Already in 1910 Grelling recognized set theory as the foundation of mathematics as a whole: "Zuerst ausgebildet als Hilfsmittel der Untersuchung bei gewissen Fragen der Analysis, hat sich die under den Händen inhres Schöpfers Georg Cantor und sein Schüler zu einer selbständigen metahmatischen Disziplin entwickelt, die heute die Grundlage der gesamten Mathematik bildet." ["In the first place developed as an auxiliary tool of the investigation of certain questions of analysis (set theory) in the hands of Cantor and his pupils (it was) developed into an independent mathematical discipline. Currently it constitutes the foundation of mathematics in its entirety" (Grelling 1910, p. 6).

[16] He also states: "For our human understanding the concept of number is more immediate than the representation of space" (Bernays 1976, p. 75).

showing that the meaning of spatial extension *intrinsically* presupposes ("builds upon") the meaning of number.

2.9.2 Interconnections Between Functional Domains

A metaphorical way to capture this state of affairs is to use an image from human memory by saying that within the meaning of space (both at the law-side and the factual side), we discover configurations *reminding* us of the core meaning of number. A key element in all metaphorical descriptions is found in the connection between *similarities* and *differences*. Whenever what is different is shown in what is similar one may speak of *analogies*. Yet we want to broaden the scope of an analogy in order to include more than what is normally accounted for in a theory of metaphor. Our first designation already achieves this goal, for whenever *differences* between entities and properties bring to expression what is *similar* between those entities or properties, we meet instances of an *analogy*.[17] Implicit in the nature of an analogy is the distinction between something *original* and something else which 'reminds' one of what is originally given but that is now encountered in a *non-original* context, i.e. within an *analogical* setting. This is exactly what we have noticed in the terms 'distance' and 'dimension'—for in both cases the use of numerical terms in a spatial context *remind* us of their original (non-spatial) quantitative meaning. In terms of the idea of an analogy one can say that there is an analogy of number on the law-side of the spatial aspect (one, two, three or more dimensions) and that there is an analogy of number at the factual side of the spatial aspect (magnitude—as the correlate of different orders of extension: in *one* dimension magnitude appears as length, in *two* dimensions it appears as area, in *three* as volume). An account of the basic position of number can now also be articulated in terms of the idea of such analogies, for since basic *numerical analogies* are presupposed within the domain of space, the original meaning of number is indeed foundational to the meaning of space.

In the previous paragraph we introduced a new word in order to refer to the domains of number and space, namely the term 'aspect'. The underlying hypothesis of this usage is found in the theory that the various aspects of reality belong to a distinct dimension which is fundamentally different from the concrete *what-ness* of (natural[18] and social) entities (such as things, plants, animals, artifacts, societal collectivities and human beings).

[17] Whenever *entities* are involved in the figurative mode of speech such designations are considered to be *metaphorical*. But as soon as similarities and differences between modal functions (as they will be explained below) are captured, these purely aspectual interrelations represent a domain of analogies distinct from metaphors. When purely intermodal connections (analogies) are metaphorically explored, an element of the entitary dimension of reality will always be present (such as it is found in the metaphor of a person being "reminded" of an original domain).

[18] The term 'natural' intends to capture the realms ('kingdoms') found in reality—things, plants and animals and they are distinct from societal realities. The latter are lived out through the social activities of human beings and the latter are guided by the *normative considerations* of the logical and post-logical aspects. Although such normatively guided actions may take the lead of any normative aspect, human beings are not qualified by any one of them. For that reason humankind cannot be subsumed in any

These concrete entities (and the processes in which they are involved) all function within the different aspects of reality. Questions about the way in which entities exist concern their *how-ness*, their mode of being. Aspects in this sense are therefore (ontic) *modes of being*. That my chair is *one* and has *four* legs reveals its function within the quantitative mode of reality; that it has a certain *shape* and *size* highlights its spatial function; that one can identify and distinguish it highlights its logical-analytical function, that it has a certain economic value demonstrates its function within the economic mode of reality, that it is beautiful or ugly brings to expression its aesthetic function, and so on.

This dimension of functions or aspects can also be designated as that of *modalities* or *modal functions*. What has already been said about the domains of number and space concern properties that may serve to define the nature of an aspect. Of course any description of modal aspects inevitably employs *metaphors* (involving entitary analogies). For example, one may say that aspects are 'points of entry' to reality, that they provide an 'angle of approach' to reality, and so on. Conversely, the modal aspects provide access to the dimension of entities—they may serve as *modes of explanation* of concrete reality.

Every aspect contains a sphere of modal (functional) laws (at its law-side); a factual side (subjected to modal laws); and a core meaning qualifying, characterizing or stamping all the structural moments discernable within an aspect (in particular also the analogical elements pointing to the meaning of other modal functions of reality). This core meaning or meaning-nucleus guarantees the uniqueness and irreducibility of every aspect and it underlies the inevitable use of primitive (= indefinable) terms by those disciplines that explore a specific modal aspect as angle of approach to reality. Some of these structural features of an aspect are captured in the sketch on the next page.

2.10 The Irreducible Meaning of Space Underlying Hilbert's Primitive Terms

Within the arithmetical aspect the factual relation between numbers can also be designated as *subject–subject relations*. The addition of numbers, the multiplication of numbers or establishing the numerical difference between numbers (subtraction) are all instance of relations between numbers as modal subjects, i.e. subjected to numerical laws. However, at the factual side of the spatial aspect there are not merely subject–subject relations (such as intersecting lines), for there are also *subject–object relations* present in this aspect. Its first manifestation is found in the idea of a *boundary*.

Already in his abstraction theory Aristotle employed the notion of a boundary (or limit)—which is intuitively immediately associated with *spatial* notions (Aristotle used the term *eschaton*). By the thirteenth century AD Thomas Aquinas accounts for a 1-dimensional line by means of a descending series of abstractions. In contradistinction to natural bodies, all mathematical figures are infinitely divisible. The Aristotelian legacy is clearly seen in his definition of a point as the *principium*

Footnote 18 continued
'kingdom'—the normative structure qualifies the temporal *Gestalt* of being human, but in itself it is not qualified by any aspect—the human being has an eternal destination.

of a line (cf. *Summa Theologica*, I,II,2), which indicates the fact that a determinate line-stretch has points at its extremities ("cuius extremitates sunt duo puncta"—*Summa Theologica*, I,85,8). This legacy returns in a somewhat more general form in the eighteenth century (the era of the *Enlightenment*). Kant remarks:

> Area is the boundary of material space, although it is itself a space, a line is a space which is the boundary of an area, a point is the boundary of a line, although still a position in space (Kant 1783, A:170).

In 1912 Poincaré discussed similar problems. Concerning the way in which geometers introduce the notion of three dimensions he says: "Usually they begin by defining surfaces as the boundaries of solids or pieces of space, lines as the boundaries of surfaces, points as the boundaries of lines" (cf. Hurewicz and Wallman 1959, p. 3). Although only related to three dimensions, Poincaré here provides us with an intuitive approach to dimension, implicitly stressing the unbreakable correlation between the law-side and the factual side in the spatial aspect:

> … if to divide a continuum it suffices to consider as cuts a certain number of elements all distinguishable from one another, we say that this continuum is of one dimension; if, on the contrary, to divide a continuum it is necessary to consider as cuts a system of elements themselves forming one or several continua, we shall say that this continuum is of several dimensions (Hurewicz and Wallman 1959, p. 3).

Before 1911 the problem of dimension was confronted with two astonishing discoveries. Cantor showed that the points of a line can be correlated one-to-one with the points of a plane, and Peano mapped an interval continuously on the whole of a square. The crucial question was whether, for example, the points of a plane could be mapped onto the points of an interval in both a continuous and one-to-one way. Such a mapping is called homeomorphic. The impossibility to establish a homeomorphic mapping between a "m-dimensional set and a $(m + 1)$-dimensional set $(h > 0)$" was solved by Lüroth for the case where $m = 3$ (Brouwer 1911, p. 161). Brouwer provided the first general proof of the invariance of the number of a dimension (see Brouwer 1911, pp. 161–165). Exploring suggestions of Poincaré, Brouwer introduced a precise (*topologically invariant*) definition of *dimension* in 1913, which was independently recreated and improved by Menger and Urysohn in 1922 (cf. Hurewicz and Wallman 1959, p. 4). Menger's formulation (still adopted by Hurewicz and Wallman) simply reads:

(a) the empty set has dimension -1,
(b) the dimension of a space is the least integer n for which every point has arbitrarily small neighborhoods whose boundaries have dimension less than n (Hurewicz and Wallman 1959, p. 4, cf. p. 24).[19]

Whereas a spatial subject is always factually extended in some dimension (such as a 1-dimensional line, a 2-dimensional area, and so on), a spatial object merely

[19] See also Brouwer 1924, p. 554. When a "species" π does not contain a continuum as part it is of dimension 0 in the Menger-Urysohn sense.

serves as a *boundary* (in a delimiting way). The boundaries of a determined line-stretch are the two points delimiting it (with the line as a 1-dimensional spatial subject). But these boundary points themselves are not extended in one dimension. Within one dimension points are therefore not spatial subjects but merely spatial objects, dependent upon the factual extension of the line. Yet a line may assume a similar delimiting role within two dimensions—for the lines delimiting an area are not themselves extended in a two dimensional sense. Likewise a surface can fulfill the role of a spatial object, namely when it delimits three dimensional spatial figures (such as a cube).

In general it can therefore be stated that whatever is a spatial subject in n dimensions is a spatial object in $n + 1$ dimensions. A point is a spatial object in one dimension (an objective numerical analogy on the factual side of the spatial aspect), and therefore a spatial subject in *no* dimension (i.e. in *zero* dimensions). In terms of the fundamental difference between a spatial subject and a spatial object, it is impossible to deduce spatial extension from spatial objects (points). It is therefore unjustifiable to see a line as a set of points. But it falls outside the scope of this presentation to highlight the circularity present in Grünbaum's attempt to argue for a consistent conception of the extended linear continuum as an aggregate of unextended elements (see Grünbaum 1952). Grünbaum did not realize that the actual infinite—or, as we prefer to call it: the *at once infinite*—depends upon a crucial *spatial* feature, namely the *spatial order of simultaneity*. In the idea of the at *once infinite* the meaning of number points towards the meaning of space in an analogical way. Every known attempt to reduce space to number employs the at once infinite—and since the latter pre-supposes the irreducibility of the spatial order of at once these attempts all turn out to be *circular* (in the sense that one can reduce space to number if and only if one assumes the irreducibility of space). This remark also applies to the ideas advanced by Carl Posy in connection with building a "continuous manifold" out of "real numbers" (see Posy 2005, 321ff).

We can now account for the three primitive terms in Hilbert's axiomatization of geometry in the context of the spatial subject–object relation. The term 'line' reflects the primary existence of a (one dimensional) spatial *subject*, the term 'point' highlights the primary existence of a (one dimensional) spatial *object* and the phrase 'lies on' accounts for the *relation* between a spatial subject and a spatial object—in other words, it highlights the *spatial subject–object relation*.

From our discussion thus far it is clear that the theory of modal aspects constitutes a key element of a non-reductionist understanding of reality.

2.11 The Theory of Modal Aspects

The theory of *modal law-spheres* first of all acknowledges the *ontic givenness* of the modal aspects. Hao Wang remarks that Gödel is very "fond of an observation that he attributes to Bernays": "That the flower has *five* petals is as much part of objective reality as that its color is *red*" (Wang 1988, p. 202). The quantitative side (aspect) of things (entities) is not a *product* of thought—at most human reflection can *explore* this given (functional) trait of reality by analyzing what is entailed in the *meaning* of multiplicity. Yet, in doing this (theoretical and non-theoretical) thought explores the *given* meaning of this quantitative aspect in various ways, normally first of all by *forming* (normally called: *creating*) **numerals** (i.e., number symbols). The simplest act of counting already explores the *ordinal meaning* of the quantitative aspect of reality. Frege correctly remarks "that counting itself rests on a one–one correlation, namely between the number-words from 1 to *n* and the objects of the set" (quoted by Dummett 1995, p. 144).

However, in the absence of a sound and thought-through distinction between the *dimension of concretely existing entities* (normally largely identified with 'physical' or "space–time existence") and the *dimension of functional modes* (aspects) of ontic reality, which cannot be observed through sensory perception, mathematicians oftentimes struggle to account for the epistemic status of their "subject matter." Perhaps the awareness for the need of acknowledging this distinct dimension of reality is best articulated in Wang's discussion of Gödel's thought. Wang discusses Gödel's ideas regarding "mathematical objects" and mentions his rejection of Kant's conception that they are 'subjective'. Gödel holds: "Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality" (quoted by Wang 1988, p. 304, cf. p. 205). To this Wang adds his support: "I am inclined to agree with Gödel, but do not know how to elaborate his assertions. I used to have trouble by the association of objective existence with having a fixed 'residence' in spacetime. But I now feel that 'an aspect of objective reality' can exist (and be 'perceived by semiperceptions') without its occupying a location in spacetime in the way physical objects do" (Wang 1988, p. 304).

Of course Wang could have referred to the important insights of Cassirer in this regard. Already in his article on Kant and modern mathematics (1907), and particularly in his influential work: *Substance and Function* (1910), Cassirer distinguishes between *entities* and *functions*. He clearly realizes that quantitative properties are not exhausted by any individual entity: "number is to be called universal not because it is contained as a fixed property in every individual, but because it represents a constant condition of judgment concerning every individual as an individual" (Cassirer 1953, p. 34). If we set aside the (neo-)Kantian undertones of this statement, Cassirer already saw something of the *modal universality* of the arithmetical aspect of reality.

Every aspect has an undefinable core (or: nuclear) meaning (also designated as the meaning-nucleus) which *qualifies* all the analogical meaning-moments within a specific aspect. These analogical moments may refer backwards to *ontically earlier* aspects (known as *retrocipations*) or forwards to *ontically later* aspects (known as *anticipations*). *Earlier* and *later* are taken in the sense of the *cosmic time-order* as it is called by Dooyeweerd. The aspects of reality are fitted in an inter-modal coherence of earlier and later. The most basic aspect is that of number (meaning-nucleus: discrete quantity), which is followed by the aspect of space (continuous extension), the kinematical aspect (core: constancy), the physical (change/energy operation), the biotic (life), the sensitive (feeling), the logical (analysis), the cultural-historical (formative control/power), the sign-mode (symbolical signification), and so on. At the factual side of each aspect there are subject–object relations (except for the numerical aspect where within which there are only subject–subject relations).

The meaning of an aspect finds expression in its coherence with other aspects (retrocipations and anticipations). *Retrocipatory analogies* are captured in the *elementary basic concepts* of a discipline.

The first challenge in an analysis of the *elementary (analogical) basic concepts* of the various academic disciplines is to identify the modal "home" or "seat" of particular terms.

Within an aspect we discerned a difference between the *order-side* (also known as the law-side) and its correlate, the *factual side* (that which is subjected to the law-side and *delimited* and *determined* by the latter). The numerical time-order of succession belongs to the law-side of the arithmetical aspect, and any *ordered* sequence of numbers appears at its factual side (think of the natural numbers in their normal succession). With the exception of the numerical aspect (which only have subject–subject relations), all the other aspects in addition also have subject–object relations at their factual side.[20]

2.12 The Impasse of Arithmeticism

It is intuitively clear that our awareness of *succession* and *multiplicity* (underlying the concept of an ordinal number and induction) makes an appeal to the quantitative aspect of reality. These terms therefore have their modal "seat" ("home") in the arithmetical aspect.

Of course it is natural that special scientists will attempt to reduce apparently primitive terms to familiar and more basic ones. But if such an attempt becomes circular, or even worse, contradictory, then it may be the case that the primitive terms involved are truly *irreducible*! Phrased differently: an attempt to define what is undefinable may end up in *antinomic reduction*.[21] Sometimes the challenge is not to get *out* of the circle, but to get *into* it(s irreducible meaning)!

In the course of our preceding discussion the following cluster of terms probably transcend the confines of the numerical aspect: *simultaneity* (at once), *completedness*, *wholeness* (totality), and the *whole-parts relation*.

The most prominent recognition of the spatial "home" of wholeness and totality is found in the thought of Bernays. He writes that it is recommendable not to distinguish the arithmetical and geometrical intuition according to the moments of the spatial and the temporal, but rather by focusing on the difference between the *discrete* and the *continuous*.[22] Being fully aware of the arithmeticistic claims of modern analysis it is all the more significant that Bernays questions the attainability of this ideal of a *complete arithmetization* of mathematics. He categorically writes:

[20] The identifiability and distinguishability of something represents its latent logical object-function. When it is identified and distinguished by a thinking subject, this analytical object-function is made patent.

[21] The classical example is Zeno's attempt to define movement in static spatial terms.

[22] "Es empfiehlt sich, die Unterscheidung von "arithmetischer" und "geometrischer" Anschauung nicht nach den Momenten des Räumlichen und Zeitlichen, sondern im Hinblick auf den Unterschied des Diskreten und Kontinuierlichen vorzunehmen" (Bernays 1976, p. 81).

> We have to concede that the classical foundation of the theory of real numbers by Cantor and Dedekind does not constitute a *complete* arithmetization of mathematics. It is anyway very doubtful whether a complete arithmetization of the idea of the continuum could be fully justified. The idea of the continuum is after all originally a geometric idea (Bernays 1976, pp. 187–188).[23]

Particularly in explaining the difference between the potential and the actual infinite the difference between *succession* and *at once* and the irreducibility of the notion of a *totality* surfaces. Hilbert introduces the difference between the potential and the actual (or: genuinely) infinite by using the example of the "totality of the numbers 1, 2, 3, 4, …" which is viewed as a unity which is given at once (completed):

> If one wants to provide a brief characterization of the new conception of infinity introduced by Cantor, one can indeed say: in analysis where the infinitely small and the infinitely large feature as limit concept, as something becoming, originating and generated, that is, as it is stated, with the potential infinite. But this is not the true infinite. We have the latter when, for example, we view the totality of the numbers 1, 2, 3, 4, … as a completed unity or when we observe the points of a line as a totality of things, given to us as completed. This kind of infinity is designated as the actual infinite.[24]

According to Lorenzen the understanding of real numbers with the aid of the actual infinite cannot camouflage its ties with space (geometry):

> The overwhelming appearance of the actual infinite in modern mathematics is therefore only understandable if one includes geometry in one's treatment. … The actual infinite contained in the modern concept of real numbers still reveals its descent (Herkunft) from geometry (Lorenzen 1968, p. 97).

Lorenzen highlights the same assumption when he explains how real numbers are accounted for in terms of the actual infinite:

> One imagines much rather the real numbers as all at once actually present—even every real number is thus represented as an infinite decimal fraction, as if the infinitely many figures (Ziffern) existed all at once (alle auf einmal existierten) (Lorenzen 1972, p. 163).

Our discussion regarding $2 + 2 = \sqrt{8}$ argued that within the quantitative aspect the order of succession (on its law-side) provides a basis for arithmetical operations

[23] "Zuzugeben ist, daß die klassische Begründung der Theorie der reellen Zahlen durch Cantor und Dedekind keine *restlose* Arithmetisierung bildet. Jedoch, es ist sehr zweifelhaft, ob eine restlose Arithmetisierung der Idee des Kontinuums voll gerecht werden kann. Die Idee des Kontinuums ist, jedenfalls ursprünglich, eine geometrische Idee.".

[24] "Will man in Kürze die neue Auffassung des Unendlichen, der Cantor Eingang verschafft hat, charakterisieren, so könnte man wohl sagen: in der Analysis haben wir es nur mit dem Unendlichkleinen und dem Unendlichengroßen als Limesbegriff, als etwas Werdendem, Entstehendem, Erzeugtem, d.h., wie man sagt, mit dem potentiellen Unendlichen zu tun. Aber das eigentlich Unendliche selbst ist dies nicht. Dieses haben wir z. B., wenn wir die Gesamtheit der Zahlen 1, 2, 3, 4, … selbst als eine fertige Einheit betrachten oder die Punkte einer Strecke als eine Gesamtheit von Dingen ansehen, die fertig vorliegt. Diese Art des Unendlichen wird als aktual unendlich bezeichnet" (Hilbert 1925, p .167).

such as addition and multiplication and their inverses and it also makes possible our basic numerical awareness of *greater* and *lesser*. The arithmetical order of succession therefore determines our most basic intuition of infinity, in the literal sense of one, another one, and so on, without an end, *endlessly*, *indefinitely*, *infinitely*. The traditional designation of this kind of infinity, known as the *potential infinite*, lacks an intuitive appeal. But when we alternatively refer to the 'successive infinite' this shortcoming is left behind. The other kind of infinity, traditionally known as the *actual infinite*, also calls for an "intuitively transparent" designation—such as the *at once infinite*. The successive infinite, presupposed in the infinite divisibility of continuity, makes possible induction, which, according to Weyl, guarantees that mathematics does not collapse into an enormous tautology (Weyl 1966, p. 86). According to Gödel non-"tautological" relations between mathematical concepts "appears above all in the circumstance that for the primitive terms of mathematics, axioms must be assumed" (Gödel 1995, pp. 320–321). In the case of finitism where the "general concept of a set is *not* admitted in mathematics proper … induction must be assumed as an axiom" (Gödel 1995, p. 321).

These modes of speech highlight the inevitability of employing terms with a *spatial descent* even when the pretention is to proceed purely in *numerical* terms. Lorenzen correctly points out that arithmetic by itself does not provide any motive for the introduction of the actual infinite (Lorenzen 1972, p. 159). The fundamental difference between arithmetic and analysis in its classical form, according to Körner, rests on the fact that the central concept of analysis, namely that of a real number, is defined with the aid of actual infinite totalities ("aktual unendlicher Gesamtheiten"—1972, p. 134). Without this supposition Cantor's proof of the non-denumerability of the real numbers collapses into denumerability. While rejecting the actual infinite, intuitionism interprets Cantor's diagonal proof of the non-denumerability of the real numbers in a constructive sense—cf. Heyting (1971, p. 40), Fraenkel et al. (1973, pp. 256, 272) and Fraenkel (1928, p. 239 note 1). However, in order to reach the conclusion of non-denumerability, every constructive interpretation falls short—simply because there does not exist a *constructive* transition from the potential to the actual infinite (cf. Wolff 1971).

It seems to be impossible to develop set theory without "borrowing" key-elements from our basic intuition of space, in particular the (order of) *at once* and its factual correlate: *wholeness/totality*. Since spatial subjects are *extended* their multiple parts exist all at once. This multiplicity is at the factual side of the spatial aspect a *retrocipation* to the meaning of number—i.e., *multiple parts* analogically reflect the meaning of number (multiplicity) within space.

Bernays did not have a theory of modal aspects at his disposal and therefore lacks the conceptual tools for articulating explicitly the intermodal connections between number and space. For example, instead of saying that the mathematical analysis of the *meaning of number* reveals an anticipation to the meaning of space, he states that the idea of the continuum is a geometrical idea which analysis expresses with an arithmetical language.[25]

[25] "Die Idee des Kontinuums ist, jedenfalls ursprünglich, eine geometrische Idee, welche durch die Analysis in arithmetischer Sprache ausgedrükt wird" (Bernays 1976, p. 74).

When, under the guidance of our theoretical (i.e., modally abstracting) insight into the meaning of the spatial order of simultaneity, the original modal meaning of the numerical time-order is disclosed (deepened), we encounter the *regulatively deepened anticipatory* idea of actual or completed infinity. Any sequence of numbers may then, directed in an anticipatory way by the spatial order of simultaneity, be considered *as if* its infinite number of elements are present as a *whole* (*totality*) *all at once*.

In this context it is noteworthy that Hao Wang informs us that Kurt Gödel speaks of sets as being "quasi-spatial" and then adds that he is not sure whether Gödel would have said the "same thing of numbers" (1988, p. 202). This mode of speech is in line with our suggestion that the undefined term "element of" employed in ZF set theory actually harbours the *totality* feature of continuity. The implication is that in an anticipatory way set theory is dependent on "something spatial"!

This also amounts to a confirmation of the unbreakable coherence between the law-side and the factual side of the numerical and the spatial aspects. The modal anticipation from the numerical time-order to the spatial time-order must therefore have its correlate at the factual side. At the factual side of the numerical aspect we first of all encounter the sequence of natural numbers (expressing the primitive meaning of numerical discreteness). Then there are the integers (keeping in mind that the term 'integer' derives from wholeness and therefore points forward to what is non-integral, namely fractions). Introducing the *dense* set of rational numbers imitates the infinite divisibility of spatial continuity. Since this divisibility embodies the successive infinite it represents, within space, a retrocipation to the numerical time-order of succession. Therefore, as an anticipation to a retrocipation, the rational numbers represent the *semi-disclosed* meaning of number.

When we employ the anticipation at the law-side of the numerical aspect to the law-side of the spatial aspect we encounter the intermodal foundation of the notion of *actual infinity*—although the basic intuitions at play here are better served by the phrases suggested above, namely the successive and the at once infinite. The fact that the at once infinite *deepens* the meaning of number requires a brief remark explaining the "as if" character of this disclosed notion of infinity. The anticipation from number to space on the law-side *determines* the multiplicity of natural numbers, integers and rational numbers which are correlated with it. Our suggestion is that under the guidance of the actual infinite these sequences of numbers are considered *as if* they are present as completed (though infinite) *wholes* or *totalities* given at once.

Remark "As if": the actual infinite as a regulative hypothesis

Vaihinger developed a whole philosophy of the "as if" (*Die Philosophie des Als Ob*), in which he tries to demonstrate that various special sciences may use, with a positive effect, certain *fictions* which in themselves are considered to be *internally antinomic*. The infinite, both in the sense of being infinitely large and infinitely small, is evaluated by Vaihinger as an example of a necessary and fruitful fiction (cf. Vaihinger 1922, 87ff, p. 530). Ludwig Fischer presents a more elaborate mathematical explanation of this notion of a *fiction*. In general he argues: "The definition of an irrational number by means of a formation rule always involves an 'endless', i.e. unfinished process. Supposing that the number is thus given, then one has to think of it as the completion (Vollendung) of this unfinished process. Only in this… the internally antinomic (in sich widerspruchsvolle) and *fictitious* character of those numbers are already founded" (Fischer 1933, pp. 113–114). Without the aid of a preceding analysis of the modal meaning of number and space, this conclusion is almost inevitable. Vaihinger and especially Fischer simply use the number concept of *uncompleted infinity* (the successive infinite) as a standard to judge the (onto-)logical status of the actual infinite. Surely, within the closed (not yet deepened) meaning of the numerical aspect, merely determined by the arithmetical time-order of *unfinished succession*, the notion of an actual infinite multiplicity indeed is *self-contradictory*.

However, the meaning intended by us for the actual infinite *transcends* the limits of this concept of number since, in a regulative way, it refers to the core meaning of the spatial aspect which (in an anticipatory sense) underlies the *hypothetical* use of the time-order of *simultaneity* (the "all" viewed as being present *at once*).

> Paul Lorenzen echoes something of this approach in his remark that the meaning of actual infinity as attached to the "all" shows the employment of a fiction—"the fiction, as if infinitely many numbers are given" (Lorenzen 1952, p. 593). In this case too, we see that the "as if" is ruled out, or at least disqualified as something *fictitious*, with an implicit appeal to the primitive meaning of number.

As long as one sticks to the notion of a *process*, one is implicitly applying the yardstick of the *successive infinite* to judge the actual infinite.

> Paul Bernays did see the essentially *hypothetical* character of the *opened up* meaning of number, without (due to the absence of an articulated analysis of the modal meaning coherence between number and space) being able to exploit it fully: "The position at which we have arrived in connection with the theory of the infinite may be seen as a kind of the philosophy of the 'as if'. Nevertheless, it distinguishes itself from the thus named philosophy of Vaihinger fundamentally by emphasizing the consistency and trustworthiness of this formation of ideas, where Vaihinger considered the demand for consistency as a prejudice …" (Bernays 1976, p. 60).

Although the deepened meaning of infinity is sometimes designated by the phrase *completed infinity*, this habit may be misleading. If *succession* and *simultaneity* are irreducible, then the idea of an infinite totality cannot simply be seen as the completion of an *infinite succession*. When Dummett refers to the classical treatment of infinite structures "as if they could be completed and then surveyed in their totality" he equates this "infinite totality" with "the entire output of an infinite process" (1978, p. 56). The idea of an infinite totality simply transcends the concept of the successive infinite.

> A remarkable ambivalence in this regard is found in the thought of Abaraham Robinson. His exploration of *infinitesimals* is based upon the meaning of the at once infinite. A number a is called *infinitesimal* (or *infinitely small*) if its

absolute value is less than *m* for all positive numbers *m* in $\Re$ ($\Re$ being the set of real numbers). According to this definition 0 is *infinitesimal*. The fact that the infinitesimal is merely the correlate of Cantor's transfinite numbers is apparent in that *r* (*not equal to* 0) is infinitesimal if and only if *r to the power minus 1* (r^{-1}) is infinite (cf. Robinson 1966, 55ff). In 1964 he holds that "infinite totalities do not exist in any sense of the word (i.e., either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally, *meaningless*." Yet he believes that mathematics should proceed as usual, "i.e., we should act *as if* infinite totalities really existed" (Robinson 1979, p. 507).

Cantor explicitly describes the *actual infinite* as a constant quantity, *firm and determined in all its parts* (Cantor 1962, p. 401). Throughout the history of Western philosophy and mathematics, all supporters of the idea of *actual infinity* implicitly or explicitly employed *some* form of the *spatial order of simultaneity*. What should have been used as an *anticipatory regulative hypothesis* (the idea of *actual infinity*), was often (since Augustine) reserved for God or an eternal being, accredited with the ability to oversee any infinite multiplicity *all at once*.

This anticipatory regulative hypothesis of actual infinity does not *cancel* the original modal meaning of number, but only *deepens* it under the guidance of theoretical thought.

The new phrases for speaking of the *potential* and *actual infinite*, namely the successive and at once infinite, already had surfaced in the disputes of the early fourteenth century concerning the infinity of God.[26]

These new expressions relate directly to our basic numerical and spatial intuitions, viz., our awareness of *succession* and *simultaneity*—and their mutual irreducibility is based upon the irreducibility of the aspects of number and space.[27]

A truly deepened and disclosed account of the real numbers cannot be given without the aid of the *at once infinite*. That this *anticipatory coherence* between number and space always functioned prominently in a deepened account of the real numbers, may be shown from many sources. It will suffice to mention only *one* in this context. But before we do that we have to return briefly to the relationship between mathematics and logic.

2.13 The Circularity Entailed in Set Theoretical Attempts to Arithmetize Continuity

The nuclear meaning of space is *indefinable*. If one tries to define the *indefinable* two equally objectionable options are open:

(i) either one ends up with a *tautology*—coherence, being connected, and so on, are all synonymous terms for *continuity*—or, even worse,

[26] Compare the expressions *infinitum successivum* and *infinitum simultaneum* (Maier 1964, pp. 77–79).

[27] Dooyeweerd did not accept the idea of the *at once infinite* (*actual infinity*) owing to the fact that he was strongly influenced by the intuitionistic mathematicians Brouwer and Weyl in this regard. Cf. Dooyeweerd, 1997-I, pp. 98–99 (footnote 1) and 1997–II, p. 340 (footnote 1).

(ii) one becomes a victim of (antinomic) *reduction*, i.e. one tries to reduce what is indefinable to something familiar but distinct.

While the idea is ancient, modern Cantorian set theory again came up with the conviction that a spatial subject such as a particular line must simply be seen as an infinite (technically, a non-denumerably infinite) set of points.

If the points which constitute the one dimensional continuity of the line were themselves to possess any extension whatsoever, it would have had the absurd implication that the continuity of every point is again constituted by smaller points than the first type, although necessarily they also would have had some extension. This argument could be continued *ad infinitum*, implying that we would have to talk of points with an ever-diminishing "size." In reality such "diminishing" points do not at all refer to real points, since they simply reflect the nature of continuous extension, which as we have seen, is *infinitely divisible*. Such points build up space out of space.

Anything which has *factual extension* has a subject-function in the spatial aspect (such as a chair) or is a modal subject in space (such as a line, a surface, and so forth). A point in space, however, is always dependent on a spatial subject since it does not itself possess any extension (see our earlier discussion).

The length, surface or volume of a point is always zero—it has none of these. If the measure of one point is zero, then any number of points would still have a zero-measure. Even a non-denumerable infinite set of points would never constitute any positive distance, since distance presupposes an extended subject.

Grünbaum has combined insights from the theory of point-sets (founded by Cantor) with general topological notions and with basic elements in modern dimension theory in order to arrive at an apparently consistent conception of the extended linear continuum as an aggregate of unextended elements (Grünbaum 1952, 288ff). From his analysis it is clear that he actually had "unextended unit point-sets" in mind and not simply a set of "unextended points" (Grünbaum 1952, p. 295). Initially he starts with a non-metrical topological description and then, later on, introduces a suitable metric normally used for Euclidean spaces (point-sets). The all-important presupposition of this analysis is the acceptance of the linear Cantorean continuum (arranged in an order of magnitude, i.e. the class of all real numbers) (cf. Grünbaum 1952, p. 296).

On the basis of certain distance axioms, the real function $d(x,y)$ (called the distance of the points x,y which have the coordinates x_i,y_i) is used to define the length of a point-set constituting a finite interval on a straight line between two fixed points (the number of this distance is its *length*). For example, the length of a finite interval (a,b) is defined as the non-negative quantity $b - a$ (disregarding the question about the interval's being closed, open, or half-open). In the limiting case of $a = b$, the interval is called "degenerate" with length zero (in this case we have a set containing a single point) (cf. Grünbaum 1952, p. 296).

Furthermore, division as an operation is only defined on sets and not on their elements, implying that the *divisibility* of finite sets consists in the formation of proper non-empty *subsets* of these (surely, the degenerate interval is indivisible by virtue of its lack of a subset of the required kind) (Grünbaum 1952, p. 301). Finally, the following two propositions are asserted and are considered to the perfectly consistent:

1. “The line and intervals in it are infinitely divisible” and
2. “The line and intervals in it are each a union of indivisible degenerate intervals” (1952, p. 301).

If we confront this analysis of Grünbaum with our characterization of the nature of the actual infinite (the at once infinite), we soon realize that his whole approach is *circular*. We have seen that, on the basis of the regulative hypothesis of the at once infinite, not only the set of real numbers but also the number of line segments having a common end point could be considered as *non-denumerable infinite totalities*. In the latter case (i.e., in the case of a group of line segments), we may identify, within the modal structure of space, a retrocipation to an anticipation (a mirror-image of the structure of the system of rational numbers). This retrocipation to an anticipation ultimately underlies Grünbaum’s statement: “the Cantorean line can be said to be already actually infinitely divided” (Grünbaum 1952, p. 300).

Seemingly, the objection that any denumerable sum of degenerate intervals (with zero-length) must have a length of zero, does not invalidate Grünbaum’s claim that a positive interval is the union of a continuum of degenerate intervals, because in the latter case we are confronted with a non-denumerable number of degenerate intervals—obviously, if we cannot enumerate them, we cannot *add* them to form their sum (for this reason, measure-theory also side-steps the mentioned objection, valid for the denumerable case). (Any attempted “addition” would leave out at least one of them.)

In this argumentation the irreducibility of the spatial time-order of simultaneity to the numerical time-order of succession is presupposed—ultimately dependent on the irreducibility of the modal meaning of space to that of number. From this it directly follows that the spatial whole-parts relation, determined by the spatial order of simultaneity, is also irreducible—explaining why the typical *totality* character of the continuum reveals an unavoidable circularity in the attempted purely arithmetical ‘definition’ of continuity. In other words, the modal meaning of space, qualified by the primitive meaning-nucleus of continuous extension (expressing itself at the law-side as a simultaneous order for extension and at the factual side as dimensionally determined extension—with or without a defined metric), not only implies that this meaning-nucleus of space is irreducible to number, but also that the spatial order of simultaneity at the law-side and the whole-parts relation at the factual side of the spatial aspect are ultimately irreducible. Therefore the attempt to reduce space to number is circular, for it has to employ the idea of the at once infinite which presupposes (in an anticipatory way) the irreducible meaning of space.

Although he did not pay attention to the law-side of the spatial aspect (obviously because he did not dispose of an articulated meaning-analysis of the structure of number and space), Paul Bernays does appreciate the irreducibility of the spatial whole-parts relation (the totality feature of spatial continuity) (Bernays 1976, p. 74).

> The property of being a totality “undeniably belongs to the geometric idea of the continuum. And it is this characteristic which resists a complete

arithmetization of the continuum" ("Und es ist auch dieser Charakter, der einer vollkommenen Arithmetisierung des Kontinuums entgegensteht—1976, p. 74).

Laugwitz realized that the real numbers, in terms of Cantor's definition of a set, are still individually distinct and in this sense 'discrete'. According to him the set concept was designed in such a way that what is continuous withdraws itself from its grip, for according to Cantor a set concerns the uniting of well-distinguished entities, implying that the discrete still rules.[28] Although this objection actually shows that Laugwitz did not understand the difference between the successive and the at once infinite properly, in its own way it could be seen as an objection to the arithmeticistic claims of modern mathematics. In this regard Laugwitz implicitly supports Bernays's deeply felt reaction against the mistaken and one-sided nature of modern arithmeticism, expressed in his words:

> The arithmetizing monism in mathematics is an arbitrary thesis. The claim that the field of investigation of mathematics purely emerges from the representation of number is not at all shown. Much rather, it is presumably the case that concepts such as a continuous curve and an area, and in particular the concepts used in topology, are not reducible to notions of number (Zahlvorstellungen) (Bernays 1976, p. 188).

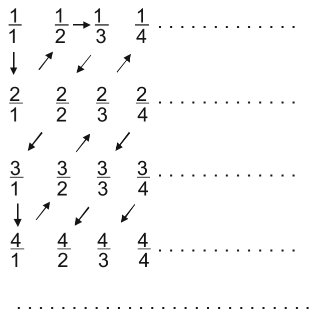
The arithmeticistic claims of set theory are circular for proving the non-denumerability of the real numbers requires (as anticipatory hypothesis) the at once infinite which in turn presupposes the irreducibility of space. Therefore, space can be reduced to number if and only if it cannot be reduced to number.

2.14 An Example of Contradicting Interpretations of An 'Exact' Mathematical Proof

2.14.1 Non-Denumerability: Cantor's Diagonal Proof

A set is called (d)enumerable when its elements can be correlated one-to-one with those of the set of natural numbers, i.e. any set the elements of which can be arranged in a natural sequence of 1, 2, 3, 4, 5, 6, … It is clear that the integers are denumerable: 0, −1, +1, −2, +2, −3, +3, … Since all rational numbers can be depicted by two integers in the form of a/b (with b 0), it is clear that they also can be denumerated. Notice the course of the arrows in the following depiction:

[28] "Der Mengenbegriff ist von vornherein so angelegt worden, daß sich das Kontinuierliche seinem Zugriff entzieht, denn es soll sich nach Cantor bei einer Menge ja handeln um eine "Zusammenfassung wohlunterschiedener Dinge …—das Diskrete herrscht" (Laugwitz 1986, p. 10). And on the next page we read: "So kommt man dazu, die Frage nach der Mächtigkeit der Menge der reellen Zahlen als "Kontinuumproblem" zu bezeichnen. In dieser Auffassung wird der Unterschied zwischen Diskretem und Kontinuierlichem verwischt: Je zwei Teilpunkte sind wohl voneinander unterschieden, aber ihre Gesamtheit soll das Kontinuum repräsentieren; dieses würde also durch Diskretes dargestellt.".



Even all algebraic numbers are denumerable. In a letter of 29 November 1873 Dedekind mentions to Cantor that he had proven that all algebraic numbers are denumerable (cf. Meschkowski 1972, p. 23). Dedekind does this by defining the height *h* of an algebraic number *x* satisfying a polynomial equation *anxn* + *an* − 1*xn* – 1 + ··· + *a*1*x* + *a*0 = 0 as follows:

$$h = n - -1 + |a_0| + |a_1| + \cdots + |a_\mathrm{n}|$$

Since the coefficients *an* are integers, only a finite number of algebraic numbers belong to each height h. Since every finite quantity is denumerable, the algebraic numbers as such are also denumerable (cf. Meschkowski 1972, p. 24).

In 1874 however Cantor proved that the real numbers are not denumerable (i.e. are non-denumerable). Only in 1890 does he provide his diagonal-proof, which we use in our explanation below (cf. Cantor 1962, pp. 278–281). A one-to-one correspondence could be established between all real numbers and the set of real numbers between 0 and 1. Furthermore, every real number in this interval can be represented as an infinite decimal fraction of the form x_n = 0.a1a2a3a4 … (numbers with two decimal representations, e.g. 0.100000… and 0.099999 … are consistently represented in the form with nines). Suppose a denumeration x_1, x_2, x_3, … exists of all the real numbers between 0 and 1, i.e. of all the real numbers in the interval $0 \leq x_n \geq 1$ (i.e. [0,1]), namely:

$$x_1 = 0.\mathrm{a}_1\,\mathrm{a}_2\,\mathrm{a}_3\ldots$$
$$x_2 = 0.\mathrm{b}_1\,\mathrm{b}_2\,\mathrm{b}_3\ldots$$
$$x_3 = 0.\mathrm{c}_1\,\mathrm{c}_2\,\mathrm{c}_3\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

If another number can be found between 0 and 1 which differs from every x_n, it would mean that every (d)enumeration of the real numbers would leave out at least one real number, which would prove that the real numbers are non-denumerable. Such a number we can construe as follows:

$$y = y_1 y_2 y_3 y_4 \ldots, \text{with } y_1 \neq 0, a_1 \text{ and } 9; y_2 \neq 0,\ b_2 \text{ and } 9;$$
$$y_3 \neq 0,\ c_3 \text{ and } 9; \text{and so forth.}$$

It is clear that y is a real number between 0 and 1 (i.e. $0 \leq y \geq 1$). The number *y* does not have two decimal representations since every decimal number in its decimal development is unequal to 0 and 9. The number is also unequal to every real number x_n since the decimal development of y in the first decimal place differs from the first decimal number x_1, in the second differs from the second decimal number of x_2 (namely x_2), and in general from the nth decimal number of x_n. It is clear from this that a denumeration of the real numbers will always exclude at least one real number ("miscount" it in the denumeration), which concludes Cantor's proof that real numbers are non-denumerable.

Also Brouwer (who identifies existence and constructibility and denies the validity of the logical principle of the excluded third with regard to the infinite) rejects the completed infinite. With this Brouwer rejects the transfinite arithmetic of Cantor. The notion of countability after all only becomes particularly relevant after the demonstration of the existence of non-denumerable cardinalities. Did Cantor not demonstrate that the set of real numbers is non-denumerable?

In Cantor's diagonal proof it is assumed that all, i.e. the at once infinite set of real numbers, are correlated one-to-one with the set of natural numbers, after which it is demonstrated that a further real number can be specified that differs from each of the counted real numbers (in at least one decimal place), from which the non-denumerability of the real numbers is concluded. The validity of this conclusion depends, however, on the acceptance of the completed infinite. Someone who recognizes only the potential infinite can never accept this conclusion, since the diagonal method then only proves that for a given constructible sequence of countable sequences (i.e. decimal expansions of real numbers) of natural numbers, yet another different countable sequence of natural numbers can be construed. Becker states this in the following way: "The diagonal method demonstrates, strictly speaking, the following: when one has a counted (law-conformative) sequence of successive numbers, a sequence of successive numbers can be calculated which differs in every place from all the previous ones" (Becker 1973, p. 161 footnote 2). In this interpretation there is no room for non-denumerability!

A mathematical proof which apparently takes an 'exact' course therefore results in conflicting conclusions depending on the presuppositions (namely completed infinity or uncompleted infinity) from which one proceeds! Fraenkel points this out emphatically:

Cantor's diagonal method does not become meaningless from this point of view, … the continuum [i.e. the real numbers] appears according to it as a set of which only a countable infinite subset can be indicated, and this by means of pre-determinable constructions (Fraenkel 1928, p. 239 footnote 1).

Whoever rejects the actual infinite (the at once infinite) cannot accept the description of real numbers given by Dedekind, Weierstrass, and Cantor.

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