Christian Philosophy and Mathematics

Prof D F M Strauss

Abstract

This presentation sets out to investigate the implications entailed in the presence of diverse trends within mathematics. The significance of Christian philosophy for mathematics requires a non-reductionist ontology which acknowledges – as an alternative to the one-sidedness of arithmeticism, geometricism and logicism – another systematic option, namely one in which the uniqueness and irreducibility of number and space (the intuitions of discreteness and continuity as Bernays prefers to designate these basic realities of mathematics) are taken seriously. At the same time it ventures to account for the unbreakable interconnectedness (mutual coherence) prevailing between the domains of discreteness and of continuity. The idea of Christian scholarship provides the basis for a complex analysis of the meaning of number and space, understood in their *ontic* sense. This task is briefly highlighted with special reference to the analogical basic concepts of mathematics, with reference to the relation between logic and arithmetic and to Dummett's analysis of intuitionism. The basic structure of the inherent circularity present in the claim that mathematics has been fully arithmetized is also succinctly described and the nature of an apparently neutral state of affairs (such as 3+4=7) is elucidated. The argumentation concludes with three remarks – including the striking confession of Hermann Weyl about the negative effect flowing from the fact that "we are less certain than ever about the ultimate foundations of (logic and) mathematics," and a similar significant remark by Fraenkel et.al. concerning the "*third foundational crisis* mathematics is still undergoing."

1. Introductory Remark

Throughout the history of mathematics various points of view found a home within the discipline of mathematics. Particularly the positivist philosohpy of science, dominant during the first part of the 20th century but in the meantime still intuitively adhered to by many special scientists, suggests that "universal reason" precludes the possibility of genuine divergent (or: conflicting) views within the so-called "exact sciences" such as mathematics (and physics).

In the philosophical legacy primarily operative within the English speaking world the term "science" is even restricted to these disciplines known as mathematics and physics. However, the difficulty with this assumptoin is that a serious scholarly account of the *nature*, *scope* and *limitis* of any scientific discipline by definition exceeds the limits of that particular discipline. The moment a mathematician wants to define or describe what mathematics is all about, something is said falling *outside* the universe of discourse of mathematics as a discipline. Suppose it is alleged that mathematics is the disicpline investigating "formal structures" (Bernays), that it is the "science of order" (Russell), that it is the "science of order in porgression" (Hamilton), or even that it is a discipline constituted by the two subdisciplines algebra and topology. Then we have to realize that all these (and many more similiar) "definitions" something is said *about* mathematics without, in any way, getting involved in *doing mathematics*. It is therefore a simple fact of philosophy of science that *talking about* mathematics is not to be equated with *being involved in the practice of mathematics*.

Naturally mathematicians may insist at this point that the only person really competent to tell us what mathematics is, is one who is acquainted with the contents of mathematics. But granting the significance of this remark does not change the fact that even when a most competent mathematician says what mathematics is what is said does not belong to the discipline of mathematics. Obviously the criterion is not: who defines mathematics, but: what is the *nature* of a definition of mathematics!

2. More preliminary questions

2.1 The demarcation problem: science and non-science

Even from the perspective of a positivist philosophy of science there are more than one discipline qualifying to receive this characterization – which immediately prompts us to investigate a more fundamental issue: what is distinctive about science as such. One cannot start with the assumption that there is a difference between the natural sciences and the humanities without first of all establishing what science as such is all about.

In a different context this question has been answered by saying that in spite of shared properties between non-scientific activities and scientific activities – such as *systematics*, *methodology*, *verification*/*falsification*, the supposed relation between a *knowing subject* and a *study object* and even *abstraction*, all scholarly endeavours share in the following uniquely disctintive feature: *modal abstraction* (see Strauss, 2001a:30 ff.). This argument is based upon the acceptance of a dimension of *ontic modes of existence* which provides a *universal functional* "grid" for the existence of any and all entities, events and social relationships. In a related context this claim is substantiated in confrontation with Frege's critique of "abstraction" and by exploring crucial insights in the thought of Cassirer, Bernays, and Gödel (particularly see Strauss, 2003:65 ff.; 70-74).

One of the most important implications of viewing "modal abstraction" as the distinctive feature of scholarly activities is that the universe ought to contain at least *two different* modal aspects. Lifting out or identifying a specific functional mode requires another one from which it is *distinguished*. For this reason every special science is always dependent upon a *more-than-special-scientific* view of reality – which is *philosophical* in nature.

2.2 Diverging trends within the "exact sciences"?

A first challenge to the positivistic belief that an exact science like mathematics is *universal*, *objective* and *neutral* would be to highlight alternative orientations within this discipline. Within the discipline of mathematics there was an on-going switch in orientation. Impressed by the possibility to use number in the description of various relationships within reality, the Pythagoreans claimed the one-sided view that everything is number. Their intention was to uphold the belief that integers and their relations (expressed in fractions) are sufficient to capture the essence of whatever there is. The first foundational crisis of mathematics surfaced when Hippasus of Metapont discovered – round about 450 B.C. – that there are relationships (such as that between any side and any diagonal of a regular pentragram) that cannot be expressed in terms of fractions. This constituted the discovery of the *irrational numbers* (see Von Frits, 1945).

The effect of this discovery was immense – not only for mathematics but also for the other disciplines and for philosophy. Greek thought in general underwent a reorientation – it now attempted to explore the meaning of space and terms with a spatial connotation in order to account for their reflections about reality. Within mathematics this shift became known as the *geometrization* of mathematics. This early development undoubtedly introduced the following foundational problem for mathematics: how does one understand the relationship between the "discrete and continuous"? Fraenkel et.al. even speak about a "gap" in this regard which have remained an "eternal spot of resistance and at the same time of overwhelming scientific importance in mathematics, philosophy, and even physics" (Fraenkel et.al., 1973:213).

These authors add a significant remark in this regard. They point out that it is not obvious which one of these two regions – "so heterogeneous in their structures and in the appropriate methods of exploring" – should be taken as starting-point. Whereas the "discrete admits an easier access to logical analysis" (explaining according to them why "the tendency of aritmetization, already underlying Zenon's paradoxes may be perceived in axiomatics of set theory"), the converse direction is also conceivable, "for intuition seems to comprehend the continuum at once," and "mainly for this reason Greek mathematics and philosophy were inclined to consider continuity to be the simpler concept" (Fraenkel et.al., 1973: 213).

Ever since Descartes introduced his analytic geometry the tendency to come to a consistent arithmetization of mathematics increasingly conquered mathematical spirits, particularly becasue the second foundational crisis of mathematics (after the independent discovery of the calculus by Leibniz and Newton) did not succeed to come to terms with the nature of the *infinite* and with *limits*.

The introduction of (irrational) real numbers remained a burden for mathematics until the so-called actual infinite effectively was made fruitful. In his Textbook of Analysis from the year 1821 the French mathematician Cauchy writes:

> "When the successive values assigned to a variable indefinitely approaches a fixed value to the extent that it eventually differs from it as little as one wishes, then this last (fixed value) can be characterized as the limit of all the others."[1]

Cauchy still thought that one can obtain an irrational (real) number with the aid of a *convergent series* of *rational* numbers, without recognizing the *circularity* in this argument. Since 1872 Cantor and Heine made it clear that the existence of irrational (real) numbers is *presupposed* in the definition of a limit. In 1883 Cantor expressly rejected this circle in the definition of irrational real numbers (Cantor, 1962:187). The eventual description of a limit still found in textbooks today was only given in 1872 by Heine, who was a student of Karl Weierstrass with Cantor (cf. Heine, 1872:178,182). However, in 1887 Cantor pointed out that the core of the ideas in Heine's article were borrowed from him (Cantor, 1962: 385). Furthermore, Cantor himself wrote an article on trigonometric series in 1872 (*Mathematische Annalen*, Volume 5) in which he gave an equivalent de-

1 "Lorsque les valuers successivement attribuées à une même variable s'approchent indéfiniment d'une valuer fixe, de manière à finir par en différer aussi peu que l'on voudra, cette dernière est appelée la limite de toutes autres" – quoted by Robinson, 1966:269.

scription of a limit with reference to convergent sequences of rational numbers (see Cantor, 1962:93).

The ultimate consequence of the mentioned urge to come to a full and complete arithmetization of mathematics is lucidly captured in the acknowledgement of Cantor, namely that he had no other choice but to employ the possibly most general concept of a purely arithmetical continuum of points.[1] This step accomplished the complete reversal of the geometrization of mathematics in Greek thought.

Unfortunately modern set theory turned out to be burdened by the troublesome presence of what Cantor called "inkosistente Vielheiten" ("inconsistent sets") (see Cantor, 1962: 447).

Zermelo introduced his axiomatization of set theory in order to avoid the derivation of "problematic" sets and Hilbert dedicated the greater part of his later mathematical life to develop a *proof* of the consistency of mathematics. But when Gödel demonstrated that in principle it is not possible to achieve this goal, Hilbert had to revert to intuitionistic methods in his proof theory ("meta-mathematics").

In this context the history of Gotllob Frege is perhaps the most striking. In 1884 he published a work on the foundations of arithmetic. After his first Volume on the basic laws of arithmetic appeared in 1893 Russell's discovery (in 1900) of the antinomous character of Cantor's set theory[2] for some time delayed the publication of the second Volume in 1903 – where he had to concede in the first sentence of the appendix that one of the corner stones of his approach had been shaken.

Close to the end of his life, in 1924/25, Frege not only reverted to a *geometrical source of knowledge*, but also explicitly rejected his initial logicist position. In a sense he completed the circle – analogous to what happened in Greek mathematics after the discovery of irrational numbers. In the case of Greek mathematics this discovery prompted the geometrization of their mathematics, and in the case of Frege the discovery of the untenability of his "Grundlagen" also inspired him to hold that mathematics as a whole actually is geometry:

1 "Somit bleibt mir nichts Anderes übrig, als mit Hilfe der in §9 definierten reellen Zahlbegriffe einen möglichst allgemeinen rein arithmetischen Begriff eines Punktkontinuums zu versuchen" (Cantor, 1962:192).

2 Russell considered the set C with sets as elements, namely all those sets A that do not contain themselves as an element. It turned out that if C is an element of itself it must conform to the condition for being an element, which stipulates that it cannot be an element of itself. Conversely, if C is not an element of itself, it obey the condition for being an element of itself.

> "So an a *priori* mode of cognition must be involved here. But this cognition does not have to flow from purely logical principles, as I originally assumed. There is the further possibility that it has a geometrical source. ... The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis – a geometrical one in fact – so that mathematics in its entirety is really geometry" (Frege, 1979: 277).

This orientation is still alive within mathematics. We have quoted Fraenkel et.el., (1973:213) regarding the basic position of continuity. More recently Longo mentions the views of René Thom and other mathematicians: "For him, as for many mathematicians of the continuum, 'the continuum precedes ontologically the discrete', for the latter is merely an 'accident coming out of the continuum background', 'a broken line' " (Longo 2001:6). He also remarks: "By contrast Leibniz and Thom considers the continuum as the original giving, central to all mathematical construction, while the discrete is only represented as a singularity, as a catastrophe" (Longo 2001:19). Of course Longo is quite aware of the fact that the set theory of Cantor and Dedekind assigns priority to notions of discreteness "and derive the mathematical continuum from the integers" (Longo 2001:19; cf. p. 20).

3. Provisional assessment

Applying the insight that the *distinctive feature* of scholarly thinking is to be found in what we called modal abstraction (i.e., the identification of some modal aspect while distinguishing it from other modes), it is clear that the history of mathematics opted at least for three different possibilities: (i) attempt exclusively to use the quantitative aspect of reality as mode of explaining the whole of mathematics – Pythagoreanism, modern set theory (Cantor, Weierstrass), and axiomatic set theory (axiomatic formalism – Zermelo, Fraenkel, Von Neumann and Ackermann); (ii) explore the logical mode as point of entry – the *logicism* of Frege, Dedekind and Russell); and (iii) the intermediate period during which the geometrical nature of mathematics was asserted, once again taken up by Frege close to the end of his life and by the mathematicians of the continuum.[1]

4. What is the meaning of Christian Scholarship?

The legacy of reformational philosophy, particularly in the thought of Dooyeweerd, proceeds from the basic Biblical conviction that within it-

1 Bernays also consistently defended the position that continuity belongs to the core meaning of space and that the modern approach to mathematical analysis of Cantor and Weierstrass did not accomplish a complete arithmetization of the 'continuum.'

self created reality does not find an ultimate or final mode of explanation. The moment a thinker attempts to pursue this path, the honour that is due to God as *Creator*, *Sustainer* and ultimate *Eschaton* of created reality is dedicated to a mere creature. The distorting effect of this inclination is manifest in all the *antinomous* "isms" discernable within all the disciplines (not only within mathematics).

The striking quotation from Kline – already mentioned in a related context dealing with "Preliminary questions on the way to a Christian Mathematics" – in connection with the "loss of certainty" in mathematics reads as follows:

> "The developments in the foundations of mathematics since 1900 are bewildering, and the present state of mathematics is anomalous and deplorable. The light of truth no longer illuminates the road to follow. In place of the unique, universally admired and universally accepted body of mathematics whose proofs, though sometimes requiring amendation, were regarded as the acme of sound reasoning, we now have conflicting approaches to mathematics" (Kline, 1980:275-276).

In opposition to all forms of reductionism, evinced in the multiplicity of "ismic" positions found within philosophy and the various scholarly disciplines, the positive contribution of the philsophical heritage handed to us in the thought of Dooyeweerd and Vollenhoven is given in their emphasis on a *non-reductionist ontology*.[1]

Looking at the history of mathematics and the dominance of an *arithmeticistic* axiomatic formalism within contemporary mathematics, the obvious observation to be made in terms of a non-reductionist ontology is the following one: Explore the option of acknowledging the uniqueness and irreducibility of very aspect inevitably involved in practising mathematics without attempting to *reduce* anyone of the modal aspects to any other aspect. Dooyeweerd claims that whenever this anti-reductionist approach is not followed, theoretical thought inescapably gets entangled in *theoretical antinomoies*. His claim is, in addition, that the logical principle of *non-contradiction* finds its foundation in the more-than-logical (cosmological) principle of the *excluded antinomy* (*principium exclusae antinomiae*) (see Dooyeweerd, 1996-II:37 ff.). A Christian attitude within the domain of scholarship, while observing the *principium exclusae antinomiae*, will attempt to avoid every instance of a one-sided deification of anything within creation. The Biblical perspective that God is Creator and that everything within creation is dependent upon the sustaining power of God,

1 Of course Dooyeweerd prefers not to use the term ontology and instead speaks about "the cosmos." For a treatment of reductionism in mathematics see Strauss, 2001,

opens they way to the life-encompassing consequences of the redemptive work of Christ, for in Him we are *in principle* liberated from the sinful inclination to search within creation for a *substitute* for God. We are in principle liberated *from* this inclination in order to be able – albeit within this dispensation always in a provisional and fallible way – to *respect* the creational diversity with the required intellectual honesty for what it is – creaturely reality in its dependence upon God.

Therefore, while respecting the uniqueness and diversity of various aspects within created reality, it should be realized that no single aspect could ever be understood in its *isolation* from all the other aspects. In fact, the *meaning of an aspect* only comes to expression in its *indissoluble coherence* with other modes – exemplified in what is designated as the modal analogies within each modal aspect, reflecting the inter-modal coherence between a specific aspect and the other aspects. These analogies point backwards or forewards to those aspects that are earlier or later within the cosmic order of aspects and are therefore accordingly are also known as *modal retrocipations* and *modal anticipations*. Within the quantitative aspect of reality no retropations are found and within the certitudinal aspect no anticipations are found – these two aspects represent the "limiting functions" underscoring the self-insufficiency of created reality.

5. What does the possibility of a Christian mathematics entail?

Clearly, if a serious attempt is made to side-step the conflicting ismic trends operative throughout the history of mathematics, the most obvious hypothesis is contained in conjecturing the following thesis:

> Accept the *uniqueness* and *irreducibility* of the various aspects of created reality, including the aspects of quantity, space, movement, the physical, the logical-analytical, and the lingual (or: sign) mode, while at the same time embarking upon a penetrating, non-reductionist analysis of the *inter-modal connections* between all these aspects.

Of course this proposal is crucially dependent upon a more articulated account of the theory of modal aspects *as such*. At the same time the incredibly rich legacy of special scientific knowledge within the domain of mathematics ought to be integrated in such an alternative approach. Yet this does not entail that such a Christian approach will have to deal with a *different reality* – simply because such an understanding already fundamentally misunderstands the Biblical perspective. The latter actually is the only life-orientation operative within world history emphasizing the unity and the goodness of creation in its entirety – by realizing that the *di-*

rectional antithesis between what is good and bad (redemption and sin) may never be identified with the structure of God's creation.

Christians and non-Christians are not living in *different worlds* and they are not doing different things – but they indeed do the same things *differently*! They share with all human beings the ability to *think*, to *discern* and to *argue*. But given the supra-theoretical Biblical starting-point of Christian scholarly reflection within the disciplines, Christians are called to take serious the demand not the absolutize anything within creation.

5.1 The ontic status of the aspects of number and space

Early modern philosophy questioned the ontic status of the aspects of reality. Descartes, for instance, claims that number and all universals are mere *modes of thought (Principles of Philosophy*, Part I, LVII). This conviction holds that only concrete entities are real and that their properties are *human constructions*. The fundamental question is: are there not, *prior* to any human intervention, construction or cognition a *given* multiplicity of *aspects* or *functions* of reality?

Hao Wang remarks that the famous mathematician, Kurt Gödel,[1] is very "fond of an observation that he attributes to Bernays":

> That the flower has five petals is as much part of objective reality as that its color is red (quoted by Wang, 1988:202).

This position suggests that the quantitative side (aspect) of things (entities) is not a *product of thought* – at most human reflection can *explore* this given (functional) trait of reality by analyzing what is entailed in the meaning of multiplicity. Yet, in doing this, theoretical and non-theoretical thought merely explores the *given* meaning of this quantitative aspect in various ways, normally first of all by forming (normally called: 'creating') *numerals* (that is, number symbols, such as '1', '2', '3', and so on). The simplest act of counting has already explored the original meaning of the quantitative aspect of reality – and it happened in a twofold way, because (i) every successive number symbol ('1', '2', '3,'etc.) is correlated with (ii) whatever is *counted*.[2]

1 At the young age of 25 Gödel astounded the mathematical world in 1931 by showing that no system of axioms is capable – merely by employing its own axioms – to demonstrating its own consistency (see Gödel, 1931). Yourgrau remarks: "Not only was truth not fully representable in a formal theory, consistency, too, could not be formally represented" (Yourgrau, 2005:68). The devastating effect of Gödel's proof is strikingly captured in the assessment of Hermann Weyl, quoted below on page .

2 Frege correctly remarks "that counting itself rests on a one-one correlation, namely between the number-words from 1 to n and the objects of the set" (quoted by Dummett, 1995:144).

The point we want to underscore is actually straightforward: both entities and the various aspects of reality truly ‘exist’, i.e. entities and aspects are both *real*, they belong to what *in fact* is given (in an *ontic* sense). What is designated as factual should therefore not be identified with human knowledge of “facts” – in an epistemic (or epistemological) sense. (The Greek word for *knowing* is *episteme*.) What we shall call the *factual side* of reality is correlated with its *law side*. What is therefore needed is a word (or the meaning-nuance of a word) capturing both, the subject-pole of the law-subject correlation, and subject-subject and the subject-object relations. When the word ‘factual’ is used for what is given in an ontic sense, prior to human cognition, then this meaning-nuance from the semantic domain of the term *factual* meets the required yardstick – therefore in this work it will be used in this sense.

Factual reality embraces both entities and aspects. Although he appraises it perhaps not as clearly, this is exactly what a prominent modern mathematician has in mind. Paul Bernays, the co-worker of the famous mathematician David Hilbert, is convinced that mathematical axiom systems are not created out of thin air, because ultimately they relate to the *nature* of the subject matter of mathematics. He therefore explicitly questions the dominant conception that ascribes reality only to entities (he does that by saying that this conception only acknowledges *one kind* of *factuality* – namely that of the ‘concrete’; Bernays, 1976:122).[1] With Cantor's theory of *transfinite numbers* in mind, Titze says that “the domain of the transfinite” cannot be “applied to reality” and therefore “merely represents [an] ideal but not meaningless construction of the imagination” (Titze, 1984: 149). This statement denies what we call the *ontic status* of the *functional modes* (or aspects) of reality. In other words, it *rejects* the second kind of ‘factuality’ Bernays has in mind, namely the **reality** of *ontic modes* or *functions*.

Another person who is on the verge of assigning an ontic status (an existence prior to human cognition) to those facets of reality with which mathematics is concerned, is Kattsoff. He also makes a plea for the acknowledgement of both physical and mathematical factuality, although he holds that “mathematical objects” are “quite different from physical objects”: “They are clearly not the sort of things that can be observed by means of the senses” (Kattsoff, 1973:30). He argues that it is through intellectual involvement that “mathematical objects” come into sight: “In analogy to physical objects which are called sensory objects because they are observed by the senses, mathematical objects may also be called intellectual

1 Note that Bernays employs the term “factual” in the sense in which we have introduced it – referring to what is given in reality prior to human cognition.

objects (or rational objects?) because they are observed by the intellect" (Kattsoff, 1973:33). Later he calls his approach "quasi-empirical" (Kattsoff, 1973:40).

Perhaps the plea for acknowledging the aspects of reality as truly existing (and not merely products of human thought) found its most impressive advocate in the thought of Gödel who introduces the idea of 'semi-perceptions' when it concerns "mathematical objects." Next to a physical causal context within which something can be 'given', Gödel refers to data of a second kind, which are open to 'semiperceptions'. Data of this second kind "cannot be associated with actions of certain things upon our sense organs" (quoted by Wang, 1988:304). In terms of our own approach these 'semiperceptions' relate to the functional aspects of reality. Gödel says:

> It by no means follows, however, [that they] are something purely subjective as Kant says. Rather they, too, may represent "*an aspect of objective reality*" (my emphasis – DS), but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality (quoted by Wang, 1988: 304).

6. Some basic building blocks

Theoretical thinking within any discipline employs *basic concepts*. Terms that have a specific *modal meaning*, i.e., terms reflecting the unique meaning of a particular aspect count amongst the *primitives* of *scientific parlance*. Since the meaning of any particular aspect reveals itself in its coherence with other aspects, these 'primitives' are inevitably encountered as soon as the inter-modal connections between different aspects are analyzed. Even when a single inter-modal connection is analyzed its theoretical analysis involves *more than one* elementry basic concept. The same applies when categorial relationships are investigated, such as the relation between law-side and factual side (traditionally: the law-subject relation), and the subject-object relation at the factual side of reality. Type concepts reflect the *typical* way in which a certain kind of entities *specifies* the universal modal meaning of the aspects (without 'individualizing' them). All disciplines apply *typical concepts* which stem from the dimension of concretely existing (natural and social) entities and events – even mathematics not excluded, because whenever a mathematician wants to lift out a combination of functional features, entitary analogies, designated by metaphors, are employed (think about Algebra where mention is made of 'groups,' 'rings,' 'ideals,' 'radicals' and so on).

In what follows below we will enter into a provisional discussion the compound nature of investigating an elementary basic concept – with special reference to Dummett's account of intuitionism – and conclude with a succinct reference to the correlation between the law-side and what is factually subjected to it – with special reference to the apparently "neutral fact " that 3+4=7.

6.1 The complex nature of forming elementary basic concepts

An important insight developed within Dooyeweerd's Christian philosophy is given in the avenue it opens up for identifying *primitive terms* and for an analysis of the way in which they are analogically employment within the various disciplines. No single scientific discipline can operate without (unavoidably) using such *elementary* basic (*analogical*) concepts.

6.1.1 The difference between logic and arithmetic one the one hand and the numerical and logical modes on the other

Given the emphasis of modern mathematics on the important role of an underlying *logic* it may seem as if the logical-analytical mode is also foundational to the meaning of number. This option is explored by Dedekind, Frege and Russell in the advancement of their *logicist* thesis that all the basic concepts of mathematics could be deduced purely from *logic*.

Dedekind commences by contemplating (the infinity) of all possible "objects of thought"[1] and Frege actually *denies* the quantitative aspect of reality by transposing it in such a way to the logical mode that it is "attached" to a *logical* concept. In 1884 he says "that the specification of a number entails an assertion related to a concept."[2] Russell is straightfor-

1 He postulates the theorem: *There are infinite sets* ("Es gibt unendliche Systeme"), and in his attempted "proof" he claims that the totality of entities present in my mindscape is infinite ("Meine Gedankenwelt, d.h. die Gesamtheit *S* aller Dinge, welche Gegenstand meines Denkens sein können, ist unendlich" (Dedekind, 1969:14).

2 "dass die Zahlangabe eine Aussage vom einem Begriffe enthalte" (Frege, 1884:59, §46).

ward in his defense of the position that mathematics and logic are *identical* (Russell, 1956:v).[1]

Yet it turned out that the quantitative meaning of number is *presupposed* by the logical-analytical mode. This insight explains at once why the most basic criticism levelled at logicism is that it did not succeed in deriving the notion of *infinity* from logic alone. Hilbert points out that in contrast to the early attempts of Frege and Dedekind he is convinced that as a precondition for the possibility of scientific knowledge certain intuitive representations and insights are indispensible and that logic alone is not sufficient.[2] Fraenkel *et al.* also affirm: "It seems, then, that the only really serious drawback in the Frege-Russell thesis is the doubtful status of InfAx,[3] according to the interpretation intended by them" (1973:186). Myhill mentions the fact that the axioms of *Principia* do not determine how many individuals there are: "the axiom of infinity, which is needed as a hypothesis for the development of mathematics in that system, is neither provable nor refutable therein, *i.e.*, is undecidable" (Myhill, 1952: 182).

In his discussion of Frege's work of 1884 (that appeared in the "Deutsche Literturzeitung," VI. Jahrgang, 1885:728-729) Cantor remarks that Frege attempts to derive the meaning of number from the configuration of the "domain of a concept" ("Umfang eines Begriffs") and then says:

> "For such a quantitative determination of the 'domain of a concept' the concepts 'number' and 'power' in a prior sense must have been available from elsewhere, and it amounts to a reversal of what is correct when an attempt is made to find a foundation for these concepts in the concept of the 'domain of a concept'."[4]

Hilbert also realized that every attempt to analyse the meaning of logic requires a simultaneous analysis of arithmetic. In his *Gesammelte Abhandlungen* Hilbert writes:

1 Intuitionism asserts the opposite of this logicist thesis, notwithstanding the fact that Heyting developed a formalization of intuitionistic mathematics. Heyting says that "every logical theorem" is "but a mathematical theorem of extreme generality; that is to say, logic is part of mathematics, and can by no means serve as a foundation for it" (1971:6).

2 "Im Gegensatz zu den früheren Bestrebungen von Frege und Dedekind erlangen wir die Überzeugung, daß als Vorbedingung für die Möglichkeit wissenschaftlicher Erkenntnis gewisse anshauliche Vorstellungen und Einsichten unentbehrlich sind und die Logik allein nicht ausreicht" – Hilbert, 1925:190.

3 InfAx = *Axiom of Infinity*.

4 "Für eine derartige quantitative Bestimmung des 'Umfang eines Begriffes' müssen aber die Begriffe 'Zahl' und 'Mächtigkeit' vorher von andere Seite her bereits gegeben sein, und es ist eine Verkehrung des Richtigen, wenn man unternimmt, die letzteren Begriffe auf den Begriff 'Umfang eines Begriffs' zu gründen" (Cantor, 1962:440).

> "Only when we analyze attentively do we realize that in presenting the laws of logic we already had to employ certain arithmetical basic concepts, for example the concept of a set and partially also the concept of number, particularly as cardinal number [Anzahl]. Here we end up in a vicious circle and in order to avoid paradoxes it is necessary to come to a partially simultaneous development of the laws of logic and arithmetic" (Hilbert, 1970:199).

Cassirer actually gives a further step. He realizes that there is something *primary* to an arithmetical unity and multilicity and compares these "primary functions" ("Urfunktionen") with the assumed logical identity and difference as necessary elements in the set concept. From the failure of the logicist attempt to deduce the *numerical* meaning of unity and multiplicity ("Verschiedenheit") from the configuration of a *logical* unity and multiplicity, the (logicist) set theoretical conception will continue to be plagued by the epistemological suspicion of a hidden circle.[1]

6.1.2 The primitive meaning number and space

Surely any analysis of the meaning of the quantitative mode of reality is confronted with the *indefinability* of its core meaning. This indefinable core meaning ought therefore to be captured by a term or a phrase that will permit finding *synonyms* for it, without claiming that thus a *definition* was found. Once again Cassirer is quite explicit in this regard. He claims that a critical analysis of knowledge, in order to side-step a regressus *in infinitum*, has to accept certain basic functions which are not capable of being "deduced" and which are not in need of a deduction.[2]

In the thought of Bernays we find a clear insight into the appropriate terms that should be associated with the basic intuition of number and space. He holds that it is recommendable not to distinguish the arithmetical and geometrical intuition according to the moments of the spatial and the temporal, but rather by focusing on the difference between the *dis-*

1 "In der Tat ist nicht einzusehen, warum man lediglich logische Identität und Verschiedenheit, die als notwendige Momente in den Mengenbegriff eingehen, als solche Urfunktionen gelten lassen und nicht auch die numerische einheit und den numersichen Unterschied von Anfang an in diesen Kreis aufnehmen will. Eine wirklich befriedigende Herleitung aus dem anderen ist auch der mengentheoretirschen Auffassung nicht gelungen, und der Verdacht eines verstckten erkenntnistheoretischen Zirkels blieb gegenüber allen Versuchen, die in dieser Richtung gemacht werden, immer bestehen" (Cassirer, 1957:73-74).

2 "Denn die kritische Analyse der Erkenntnis wird, wenn man nicht einen regressus in infinitum annehmen will, immer bei gewissen Urfunktionen Halt machen müssen, die einer eigentlichen 'Ableitung' weder fähig noch bedürftig sind" (1957:73).

crete and the *continuous*.[1] Being fully aware of the arithmeticistic claims of modern analysis it is all the more significant that Bernays questions the attainability of the ideal of a *complete arithmetization* of mathematics. He unambiguously writes:

> "We have to concede that the classical foundation of the theory of real numbers by Cantor and Dedekind does not constitute a *complete* arithmetization of mathematics. It is anyway very doubtful whether a complete arithmetization of the idea of the continuum could be fully justified. The idea of the continuum is after all originally a geometric idea" (Bernays, 1976:187-188).[2]

6.1.3 The complexities entailed in analyzing the meaning of number

Keeping in mind that the notion of continuity (with the whole-parts relation entailed in it – see Strauss, 2002) derives from the primitive meaning of *space*, one can trace numerous examples of mathematical analyses employing spatial terms in their reflection on the meaning of number. In general an analysis of the meaning of number constantly employs terms having their "modal seat" in other aspects of reality. Even if a mathemtical orientation does not acknowledge the so-called "actual infinite" as it is employed in Cantor's set theory and in modern variants of axiomatic set theory, it still turns out to be the case that terms with a "spatial descent" are employed in its analysis of the meaning of number.

6.1.4 Dummett's account of intuitionism

For example, when Dummett explains the intuitionistic notion of infinity a striking *dialectics* surfaces. This is the case because he at once – without being critically aware of it – both *uses* and *discards* notions that are essentially spatial. He mentions the fact that infinite sequences (whether determined by a law or not), is regarded as *intensional* in character (Dummett, 1978:63). Already in 1919 Weyl advocates this notion by claiming that a certain property determines a set. This statement is equivalent to the statement that the meaning ("Sinn") of a concept is logically prior to its extension ("Umfang"). To support this view, references to Fichte and Husserl are given (cf. Weyl, 1919:86.) This implies that the truth of an *extensional* statement about an infinite sequence can only be established on the basis "of some *finite* (I am emphasizing – DS) amount

1 "Es empfiehlt sich, die Unterscheidung von "arithmetischer" und "geometrischer" Anschauung nicht nach den Momenten des Räumlichen und Zeitlichen, sondern im Hinblick auf den Unterschied des Diskreten und Kontinuierlichen vorzunehmen" (Bernays, 1976:81).

2 "Zuzugeben ist, daß die klassische Begründung der Theorie der reellen Zahlen durch Cantor und Dedekind keine *restlose* Arithmetisierung bildet. Jedoch, es ist sehr zweifelhaft, ob eine restlose Arithmetisierung der Idee des Kontinuums voll gerecht werden kann. Die Idee des Kontinuums ist, jedenfalls ursprünglich, eine geometrische Idee."

of information about it which can be acquired at some time" (Dummett, 1978:63).

As long as one sticks to the finite case, no objection is intuitionistically raised against the (implicit) use of notions with a *spatial descent*. However, as soon as the *infinite* enters the scene, intuitionism does not allow anything transcending the restricted use of the *potential infinite* (oriented to the *numerical time-order of succession*). Nevertheless, contrary to this consistent defence of a *restricted* meaning of infinity, even Dummett's own description of the intuitionistic analysis of the meaning of number frequently employs terms which make an appeal to the *original meaning of space*. Without any hesitation he speaks about "infinite *totalities* (I am italicizing – DS) of mathematical objects". On the next page the expression "infinite *domain*" is used as a substitute for "infinite totality" (cf. similar usages – Dummett, 1978:22, 24, 57, 58, 59, 63 and so on). Sometimes the phrase "infinite *structures*" is used (Dummett, 1978:56, 62).

The persistent undertone of this almost excessive use of terms like *totality*, *domain* and *structure* in order to describe the infinite, is of course that one should keep in mind that according to this exposition of Dummett any infinite structure is not something given with a set of "completed objects" (cf. Dummett, 1978:62). This is the mistake of platonism (Dummett, 1978:62). In other words, any *static* notion of infinity – which inevitably (as we will presently argue, reflects the *spatial time-order of simultaneity* in an *anticipatory way*), is unacceptable to intuitionism. The time-honored legacy which attributed the static meaning of space with timelessness is mentioned by Dummett with reference to the predicate "is true", which, on a platonistic view, "attaches timelessly to any mathematical statement to which it attaches at all" (Dummett, 1978:18).

This way of addressing the issues inevitably ends up in a dialectical description of infinity. One cannot have it both ways: on the one hand stressing the infinite as an *unfinished process*, and on the other hand referring to it as an *infinite domain* (*totality*, *structure*). Eventually Dummett had to give priority to the first (*restricted*) notion of infinity in order to do justice to the intuitionistic conception. However, he did that without abolishing the expression "infinite totality", since he simply tried to interpret it from the perspective of an unfinished process:

> "The fact that an infinite totality, such as that of the natural numbers, is understood as 'in process' comes out in the interpretation of quantification over such a totality. An infinite sequence being, unlike a natural number, an object itself in process of growth, its uncompleted character must come out in the way statements about any one such sequence are interpreted" (Dummett, 1978:63).

From this quotation we clearly see that the mentioned dialectic is actually *intensified*, because the restricted meaning of infinity (as something being "in process") is "saved" with the aid of a potentially infinite interpretation of the meaning of quantification, while still, at the same time, this meaning is ascribed to quantification *over an infinite totality*![1]

6.2 The petitio principii entalied in the attempt to arithmetize mathematics

Although axiomatic set theory proceeds under the flag of being fully arithmeticistic, it does not realize that its entire analysis of the "continuum" – interpreted as an analysis of the real numbers – inherently depends on "borrowing" something crucial from our *spatial* intuition, namely the awareness of "at once" (*simultaneity*) and the already mentioned feature of a totality (a whole with its parts). Zermelo-Fraenkel set theory accepts the primitive binary predicate designated as the *membership* relation. This move only apparently conceals any connection with our spatial intuition, for the moment in which we set out to investigate what is at stake, is is clear that the undefined status of the term 'set' (or, alternatively, the 'membership relation') borrows the two above-mentioned key features from the spatial mode, namely *simultaniety* and the *whole-parts* relation.

Therefore mathematical set theory in fact ought to be seen as a *spatially deepened theory of number*. In this context it is noteworthy that Hao Wang informs us that Kurt Gödel speaks of *sets* as being "*quasi-spatial*" and then adds that the remark that he is not sure whether Gödel would have said the "same thing of numbers" (Wang, 1988:202)!

Particularly in confrontation with the dominant claim that mathematics has been *arithmetized* completely, these insights should be embedded within the context of an inter-modal understanding of the meaning of number and space. Without explaining this point in a technical way, the inherent circularity entailed in this whole position could be highlighted on the basis of the distinction between what has been designated as the *successive infinite* (traditionally known as the potential infinite) and the at *once infinite* (tradtionally: the actual infinite). The introduction of these phrases took into account the rich legacy of philosophical and mathematical reflections on the nature of infinity. By "locating" the interconnections between these two kinds of infinity relative to the respective meanings of number and space the said *attempted arithmetization* of mathematics

1 Although Heyting is critically aware of the fact that on the intuitionistic standpoint difficulties "arise only where the totality of integers is involved," he nonetheless in the same paragraph without hesitation speaks about the "domain" of rational numbers (1971:14).

stands and falls with the acceptance of *non-denumerable sets* – and it can be shown that the only basis upon which the latter could be introduced is by employing the idea of the at once infnite. But in order to employ the *at once infinite* one has to account for the *theoretical deepening* of the primitive numerical intuition of succession in its anticipation to the spatial meaning of simultaneity (at once) underlying the (regulative) hypothesis of viewing successively infinite sequences *as if* all their elements are present *at once*. Therefore, implicitly or explicitly, the use of the at *once infinite* has to make an appeal to the meaning of space – i.e., to the spatial (time-order) of at once (simultaneity), entailing the feature of *totality* which is irreducible to *succession*. Consequently, spatial continuity could be reduced to number if and only if its irreducibility is assumed (in the inevitable acceptance of the at once infinite).[1]

6.3 Mathematics does not deal with "pure facts"

A reference to the "fact" that 2x2=4 is often used to argue against the possibility of Christian scholarship in general (and the possibility of a Christian mathematics in particular). In order to highlight a few basic perspectives in this regard it is convenient to change the "fact" under consideration from 2x2=4 to 3+4=7.

Suppose now that it is asserted: 3+4=5. Although one may think that the assertion 3+4=5 is false, a different interpretation may be explored. Imagine a child first walking 3 miles to the north and afterwards 4 miles to the east and then assess how far this child is away from its point of departure.

1 Although Paul Bernays, the co-worker of the foremost mathematician of the 20th century, David Hilbert, and the author of a distinct variant of modern axiomatic set theory, did not develop the necessary theoretical distinctions advanced in this account of the (inter-modal) meaning of the *at once infinite* (actual infinity), he does have a clear understanding of the *futility* of arithmeticistic claims. He writes: "It should be conceded that the classical foundation of the theory of real numbers by Cantor and Dedekind does not constitute a complete arithmetization. ... The arithmetizing monism in mathematics is an arbitrary thesis. The claim that the field of investigation of mathematics purely emerges from the representation of number is not at all shown. Much rather, it is presumably the case that concepts such as a continuous curve and an area, and in particular the concepts used in topology, are not reducible to notions of number (Zahlvorstellungen)" (Bernays, 1976:187-188) ["Die hier gewonnenen Ergebnisse wird man auch dann würdigen, wenn man nicht der Meinung ist, daß die üblichen Methoden der klassischen Analysis durch andere ersetzt werden sollen. Zuzugeben ist, daß die klassische Begründung der Theorie der reellen Zahlen durch Cantor und Dedekind keine restlose Arithmetisierung bildet. Jedoch, es ist sehr zweifelhaft, ob eine restlose Arithmetisierung der Idee des Kontinuums voll gerecht werden kann. Die Idee des Kontinuums ist, jedenfalls ursprünglich, eine geometrische Idee. Der arithmetisierende Monismus in der Mathematik ist eine willkürliche These. Daß die mathematische Gegenständlichkeit lediglich aus der Zahlenvorstellung erwächst, ist keineswegs erwiesen. Vielmehr Iassen sich vermutlich Begriffe wie diejenigen der stetigen Kurve und der Fläche, die ja insbesondere in der Topologie zur Entfaltung kommen, nicht auf die Zahlvorstellungen zurückführen".]

The obvious answer is: 5 miles! What is here established is a *geometrical fact* – it concerns a *vector sum* (where a *vector* is defined by two properties: *distance*[1] and *direction*). Naturally establishing the *numerical fact* 3+4=7 differs from stating the *geometrical fact* 3+4=5. Since the latter is a *geometrical* fact, it is actually required that this vector sum is written with pointed arrows above the '3', '4' and '5' – in order to highlight the difference between numerical and geometrical addition.

This example demonstrates that there are no "brute" facts in mathematics. Facts are always *ordered*, i.e., they are *law-conformative* (and in the case of human beings norm-conformative or antinormative). In other words, without an implicit or explicit appeal to God's *law-order* every possible factuality disappears into nothingness, because whatever is correlated with God's law is always determined and delimited by the latter. At the law-side of the numerical aspect we discern first of all the *numerical time-order of succession* and this time-order not only provides the foundation for our most basic understanding of infinity – literally endlessness: one, another, one and so on, without an end – but also underlies the operations of addition, multiplication, subtraction and division found at the law-side of the arithmetical aspect. Whereas the numbers '3', '4' and '7' are appearing at the *factual side* of the numerical aspect, addition (+) appears at its law-side – demonstrating in a striking way that the relation established between these three numbers is determined by the numerical time-order of succession. Therefore, the (numerical) "fact" that 3+4=7 clearly is not a "brute" fact, but a *lawful* fact.

Note that the above-mentioned possibility to distinguish between a(n ordered) numerical fact (sum) and a(n ordered) geometrical fact (a vector sum) presupposes an *order-difference* between these two aspects of reality. Consequently, the mere reference to two kinds of "facts" (law-conformative states of affairs) confronts us at once with the uniqueness of *number* and *space* – theoretically accounted for by acknowledging the sphere-sovereignty of each one of these aspects while holding on to their mutual connectedness exemplified in their respective retrocipatory and anticipatory analogies. As we have seen, this perspective is absent in the entire history of mathematics. Therefore, from the "innocent" fact that 3+4=7 we are now caught up in the fluctuating trends within the history of mathematics and confronted with the dominant schools in modern mathematics.

1 "Distance," at the factual side of the spatial aspect, analogically reflects the original meaning of number – it is the "measure" of one-dimensional extension.

In stead of absolutizing either the numerical or the spatial aspect, mathematics ought to explore the third alternative suggested by a Biblically inspired non-reductionistic ontology aiming at avoiding (as Roy Clouser correctly emphasizes – see Clouser, 2005) the deification of something within creation. That is to say accepting both the uniqueness and irreducibility and the mutual coherence between number and space.

7. Concluding remarks

Our foregoing observations explored a few implications of the way in which an analysis inspired by the conjecture regarding the irreducibility and mutual coherence between number and space indeed penetrates into core issues within the discipline of mathematicians. At once it also underscores the significance of ackknowledging the conditioning role of God's order for creation. The idea of God's creation order indeed ought to continue to form a key element in the further development of reformational philosophy.

Three final remarks ought to be made:

(i) Highlighting the inner circularity entailed in the arithmeticistic claim of modern axiomatic set theory on the basis of our brief account of the inter-modal meaning of number and space, is analogous to the program of alternative research strategies in modern mathematics, such as *intuitionism* and *axiomatic formalism*. The former explored the semi-dsclosed meaning of number (thus restricting itself to the system of real numbers approached merely in terms of the successivee infinite), whereas the latter accepted as starting-point the spaitally deepened meaning of number without realizing that its reductionistic intention – viewing its entire operation in purely arithmeticistic terms – entails a vicious circle.

(ii) Although our argumentation has shown that a Biblical starting-point, sensitive to avoid reductionist approaches in mathematics, opens up new vistas in this regard, mathematical expertise is required in order to explore these rudimentary points of departure further.

(iii) It should be clear, however, that the possibility of a Christian mathematics does not have to discard the rich legacy of mathematics, but at the same time it also teaches us that mathematics is just as little as any other discipline *neutral*. The decisive ‘paradigm’ operative in the actual work of a mathematician indeed ‘colours’ what he or she does and may even direct (or: re-drict) his or her research into fields initially not contemplated. Hermann Weyl, one of the exceptionally prominent mathematicians of the 20th century, aptly states in 1946: “From this history one thing should be clear: we are less certain than ever about the ultimate foundations of (logic and) mathematics. Like everybody and everything in the world, we have our 'crisis.' We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable, practical influence on my mathematical life: it directed my interests to fields I considered relatively 'safe,' and has been a constant drain on the enthusiasm and determination with which I pursued my research work. This experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavors mean in the context of man's whole caring and knowing, suffering and creative existence in the world" (Weyl, 1946:13). This remark is echoed almost thirty years later when the second edition of Fraenkel's work on the Foundations of Set Theory states (in connection with a claim made by Poincaré at the second International Congress of Mathematicians, held in 1900 at Paris, namely that today "there remain in analysis only integers and finite or infinite systems of integers. ... Mathematics ... has been arithmetized ... We may say today that absolute rigor has been obtained"): "Ironically enough, at the same time that Poincaré made his proud claim, it had already turned out that the theory of the 'infinite systems of integers' – nothing else but a part of set theory – was very far from having obtained absolute security of foundations. More than the mere appearacne of antinomies in the basis of set theory, and thereby of analysis, it is the fact that the various attempts to overcome these antinomies, to be dealt with in the subsequent chapters, revealed a far-going and surprising divergence of opinions and conceptions on the most fundamental mathematical notions, such as set and number themselves, which induces us to speak of the third foundational crisis mathematics is still undergoing" (Fraenkel et.al., 1973:14)

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