IS *INFINITY* PURELY ARITHMETICAL IN NATURE?

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ABSTRACT

In this article we highlight some of the main contours of the urge towards the infinite in order to focus on the twofold role of infinity in mathematics. Our brief discussion of the discovery of the whole-parts relation explains the switch from infinity as endlessness to infinity turned 'inwards', evinced in the infinite divisibility of (spatial) continuity. The traditional Aristotelian distinction between the potential infinite and the actual infinite constitutes the background of our subsequent analysis which touches upon Zeno's paradoxes and Aristotle's objections to the actual infinite. Since Descartes mathematicians increasingly reverted the relation between the potential infinite and the actual infinite by considering the latter as the basis of the former. The historical dominance of the potential infinite was eventually challenged by Cantor in his transfinite arithmetic. Weyl even portrayed mathematics as the science of the infinite. However, this view prompts us to analyse in more detail what the difference between the potential infinite and the actual infinite really is. Cantor's own definition of these two kinds of infinity serves as the starting-point of our ensuing analysis. It is argued that set theory (while employing the actual infinite) crucially depends upon 'borrowing' (imitating) key features from space, namely the just-mentioned whole-parts relation and the spatial (time) order of simultaneity (at once). A spatially deepened account of the nature of real numbers has to consider them as being present at once. Attention is also given to the objections raised by Paul Bernays, the co-worker of David Hilbert, regarding the assumed arithmetization of modern mathematics. Bernays argues that it is the totality character of continuity (which is originally a *geometrical* notion) resisting a complete airthmetization of mathematics. It is striking that the spatial feature of wholeness receives opposing interpretations in the thought of Bernays and Brouwer. The former explores the *totality character* of the continuum whereas the latter focuses on the *whole-parts* relation. Ultimately the impossibility to articulate the nature of the at once infinite without (implicitly or explicitly) exploring key elements of space therefore uproots the claims of arithmeticism. Although the potential infinite is purely arithmetical in nature, the actual infinite is not, because no single account of it succeeded in avoiding the above-mentioned key spatial characteristics. Lorenzen aptly points out that arithmetic provides no motif for introducing the at once infinite. Therefore the question posed in the title of this article, namely: "Is infinity purely arithmetical in nature?" should be answered in a twofold way: (i) The potential infinite (successive infinite) is a purely arithmetical concept, whereas (ii) the actual infinite (at once infinite) is not purely numerical in nature. Some of the key elements of the argument is captured in the Figure inserted in paragraph 22.

1. THE URGE TOWARDS THE INFINITE

That human beings are *finite* in nature might have contributed to the enduring urge towards an understanding of the *infinite*. Already in early Greek philosophy we meet a philosopher, Anaximander, who designates the origin (*archē*) as the infinite-unbounded (*apeiron*). As origin the *apeiron* is without *ageing* (Diels-Kranz 1960, B. Fr.2) and without *death* and *decay* (B Fr.3).

Amidst the temporality of human life the urge towards the infinite was also accompanied by notions of incorruptibility and eternity, while the latter, *eternity*, became associated with *timelessness* and the (timeless) *present*. In addition the notion of infinity surfaced as well, particularly in early developments within mathematics. The school of Parmenides and the thought-world of the Pythagoreans come to mind. Zeno, a pupil of Parmenides, articulates a number of arguments against multiplicity and movement, subsequently known as Zeno's paradoxes: the flying arrow, Achilles en the tortoise, and the bisection paradox.

2. MOTION, MULTIPLICITY AND THE WHOLE-PARTS RELATION

Contemplate for a moment the concise formulation of B Fr. 4: "Something moving neither moves in the space it occupies, nor in the space it does not occupy". In this Fragment Zeno commences by first of all granting the reality of movement: "Something moving …". But then he proceeds by showing that it is impossible for such a thing to move at all.

For the idea of infinity the previous Fragment is important, for it may be appreciated as the first articulation of the *whole-parts relation* and what is entailed in it: the *infinite divisibility* of something continuous. The third Fragment toggles between the two sides of this relation, for first of all it argues from the parts to the whole and then from the whole to the parts. The first section reads:

> When multiplicity exists, then necessarily only as many (things) exist as what are actually there, no more and no less. When there however are as many as what exist, then it (the number thereof) must be limited.

While commencing with the same opening statement the second part reaches an opposite conclusion:

> When multiplicity exists, then that which exists (the number thereof) is unlimited. Because continually other ones exist in between those which exist and again others between these. Thus that which is (the number thereof) is unlimited.

If the multiplicity of the initial comment refers to the many parts of the universe, then their sum-total is limited (*finite*). Alternatively, if one starts by subdividing the whole, then it would be possible to proceed endlessly, for there will always be more in-between the extremities of every part.

3. THE INFINITE TURNED INWARDS

Clearly, the initial basic acquaintance with infinity is given in the awareness of an endless succession. Any succession of numbers, such as 1, 2, 3, …could be extended indefinitely, beyond all finite boundaries, for this sequence of numbers is literally without an end, endless, **infinite**. Yet, as it is shown in Zeno's B Fr.3, dividing a continuous whole turns the successive infinite *inwards*. In fact, every divided part appears to be a new whole susceptible to being divided in its own right, suggesting that the divisibility of any continuous whole entails that every part is also endlessly divisible. This justifies the view held by Aristotle, namely that “everything continuous is divisible into divisibles that are infinitely divisible” (Aristotle, *Physics* 231b15-16; Aristotle 2001:317).

The connection between infinity and succession is captured by Aristotle when he states that

> a thing may be infinite in respect of addition or of subtraction, or both. The infinite cannot be a separate, independent thing. For if it is neither a spatial magnitude nor a plurality, but infinity itself is its substance and not an accident of it, it will be indivisible; for the divisible is either magnitude or plurality. But if indivisible, it is not infinite (Aristotle *Metaph*. 1066a35-1066b4; Aristotle 2001:865).

4. THE PYTHAGOREANS: EVERY THING IS NUMBER

From what Aristotle holds it is clear that our original understanding of an endless succession was enriched through the discovery of the whole-part relation, for now we have an endless succession of *divisions*. In other words, instead of growing beyond every stage reached, i.e. being “infinite in respect of addition”, successive divisions turn the infinite inwards, manifested in the infinite divisibility of the world as a whole. The Pythagoreans explored the key position of number in their arithmeticist statement: “everything is number” (Thesleff 1970:82). Arithmetizing also musical consonants generated the general claim that if two things are related as two numbers then they are actually concealed numbers themselves (Scholz and Hasse 1928:6).

5. ZENO'S PARADOXES: POTENTIAL INFINITY AND AT ACTUAL INFINITY

At this point we have to return to Zeno's paradoxes, because Aristotle attempted to resolve them by introducing a distinction between the *potential infinite* and the *actual infinite*. This distinction is applied to the impossibility to move from point A to point B. In order to accomplish this journey the first half of the distance must be completed, then half of the remaining half, and so on endlessly, *ad infinitum*. Zeno infers that an infinite number of spatial sub-intervals must be crossed in order to traverse the distance from A to B, something impossible within a finite interval of time. According to Aristotle infinity is merely a *potentiality* that cannot be realized in *actuality* – and Aristotle's rejects the *actual infinite* on two grounds (cf. *Physica*, 204a20ff., *Metaph*., 1066b11ff., and *Metaph*., 1084 a 1ff. – cf. Aristotle 2001:260-261, 866, 904).

(i) If the actually infinite consists of parts then these parts must themselves be actually infinite, which would imply the absurdity that the whole is no longer larger than a part; and

(ii) If it consists of finite parts, this would imply the impossibility that the infinite can be counted, or there would have to be transfinite (cardinal) numbers which are neither odd nor even.[1]

[1] Later on Georg Cantor, in his transfinite arithmetic, has shown that the smallest transfinite ordinal number, omega (ω), unites four apparently contradictory features [ω indicates the set of natural

6. A FEW MEDIEVAL LINES

The subsequent development of the idea of infinity, particularly during the *Middle Ages* and early *Modernity*, further explored its meaning. Plotinus characterizes the opposing poles of matter and form both as being *infinite*. It applies to the One as the origin of everything and to matter (compare the *Enneads* II,4,4; II,4,10; II,4,15; VI,7,32). The term *infinite* is nonetheless used in a dialectically opposed manner with regard to the *One* and (formless) *matter*: as a permanent substratum matter *receives* form while the matterless One *gives* form (*En*. VI,7,17). This re-appreciation is related to Plotinus's view of infinity as the timeless present (the whole *En*.III,7), which simultaneously exerted a considerable influence on the conceptions regarding infinity of Boethius, Augustine (see his *Confessiones* XI,11,13; and *De Trinitate* XII,14), Thomas Aquinas (*Summa Theologica* I,10) and Schilder (1948:61).

Augustine went further than Plotinus for he believes that we should not use our lacking understanding of infinity as a measure for God, since God in his omniscience understands every infinity, even the completed infinitude of all numbers, without any passage of time, at once, without before and after. God therefore also knows his own completed infinite being. But creation is finite. By the end of the *Middle Ages* and the beginning of the *Modern Era* Cusanus altered this view for he holds that God is *actually infinite* whereas reality is only *endless* (potentially infinite). According to him the infinite line is simultaneously a triangle, circle and sphere (*De Docta Ignorantia*, I,13-17). Cusanus also believed that the actual infinity of God justifies saying that He is everything and nothing at all, for he is for example the largest and the smallest (*De Docta Ignorantia*, I,5). Moreover, all contradictions are resolved in God (as *coincidentia oppositorum*) (*De Docta Ignorantia*, I,22; *De Coniecturis* II,1 and II,2).

7. DESCARTES: TURNING THE CLASSICAL VIEW UPSIDE DOWN

Descartes turns the classical view on its head with his view that the infinite is complete and the finite incomplete, so that the finite should actually be referred to as the non-infinite. Since Spinoza identified God with nature (*Deus sive natura*), he also saw the universe as a completed infinity.

Galileo discussed the remarkable relation between squares and all (natural) numbers in a dialogue published in March 1638. The natural numbers include square numbers (like 1, 4, 9, 16, 25...). Now it seems as if the combination of all (natural) numbers, i.e. square and non-square numbers taken together, are certainly more than the square numbers on their own. From 1-100 there are only ten squares ($100 = 10^2$). That means that up to 100 only one tenth are squares; from 1-10000 there are only 100 squares, i.e. 100/10000 = one hundredth; from 1-1000000 there are only 1000 squares, i.e. one thousandth, and so forth. Now revert the question by asking *how many* squares exist? Surely every square has a root, which entails that

numbers in their natural order: 1, 2, 3, 4, 5, 6, ...] – which provides an answer to Aristotle's second objection. The remarkable fact is that this transfinite ordinal number ω is even (namely $2.\ \omega$), as well as uneven (namely $1 + 2.\ \omega$) and simultaneously ω is neither even (namely $\neq \omega.2$), nor uneven (namely $\neq \omega.2 + 1$) (cf. Cantor, 1962:178-179). According to Cusanus, God, as the *actual infinite*, is the union of all opposites: the *coincidentia oppositorum* – showing that he also discerned an essential characteristic of the at once infinite. See the text below.

there are as many squares as there are square roots. Then, however, there are just as many squares as there are numbers!

1^2	2^2	3^2	4^2..........
1	2	3	4...........

8. EXPLORING THE AT ONCE INFINITE: BOLZANO AND CANTOR

Bernard Bolzano built on this insight in his posthumous work on paradoxes of the infinite. He points out that in the case of an infinite set the whole set could be mapped element by element with a true subset of the original set. In terms of the example mentioned above it means that the number 1 could be correlated with 1^2, 2 with 2^2, 3 with 3^2, and so on. Clearly, the set of squares is a subset of the set of natural numbers (see Bolzano 1920:27ff. – §20). Bolzano highlights the fact that in the case of two infinite sets the whole set is equivalent to a part of it – i.e. to a subset of the original set (Bolzano 1851:28). This state of affairs is in conflict with Aristotle's above-mentioned view that if the actual infinite exists the whole must be equal to a part, contradicting his own conviction that the whole is always *greater* than a part. Aristotle's objection turns out to be defining one feature of infinite sets.

Yet Becker points out that Aristotle's view, namely that infinity (as well as continuity) only exists as a potentiality without any actuality and that it therefore remains unfinished (not completed) was only challenged when Georg Cantor, developed his "Mengenlehre" [set theory] in the second half of the 19th century. This theory not only considered actually infinite multiplicities but at once also transcended the Aristotelian basic conception of infinity (and continuity) which up till Cantor remained the unchallenged general conception of all mathematicians (if not all philosophers as well).[2]

9. WEYL: MATHEMATICS AS THE SCIENCE OF THE INFINITE

Yet defining mathematics as set theory immediately opened the way to reflect more extensively on the difference between *uncompleted infinity* and *completed infinity*. Hermann Weyl makes a striking remark in this regard: "If in conclusion one would want to provide a brief catch-word which would capture the vital core of mathematics, one would be able to say: it is the science of the infinite". Add to this the words of arguably the greatest mathematician of the 20th century, David Hilbert:

> The infinite has moved the human mind like no other question since the earliest times; the infinite has brought about mental stimulus and fruitfulness like virtually no other idea; the infinite however needs clarification like no other concept (Hilbert 1925:163).

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[2] "Die entscheidende Erkenntnis des Aristoteles war, dass Unendlichkeit wie Kontinuität nur in der Potenz existieren, also keine eigentliche Aktualität besitzen und daher stets unvollendet bleiben. Bis auf Georg Cantor, der in der 2.Hälfte des 19.Jahrhunderts dieser These mit seiner Mengenlehre entgegentrat, in der er aktual unendliche Mannigfalgtigkeiten betrachtete, ist die aristotelische Grundkonzeption von Unendlichkeit und Kontinuität das niemals angefochtene Gemeingut aller Mathematiker (wenn auch nicht aller Philosophen) geblieben" (Becker 1964:69).

As soon as one reflects on the nature of the infinite the challenge arises to explain how we have to understand the difference between die potential infinite and the actual infinite.

10. DISTINGUISHING BETWEEN THE POTENTIAL INFINITE AND THE AT ONCE INFINITE

Where Fraenkel explains the nature of infinity in the third edition of his *Introduction to Set Theory* he commences by referring to values (*Größen*) growing beyond all finite magnitudes or shrinking below all finite magnitudes by approximating Zero.[3] There are no limits to these increasing or decreasing values. In every phase of this process, regardless of how long and how far it is carried through, these magnitudes have a definite finite value or a value differing from Zero. In this case, employing the term 'infinite' merely concerns, as Gauss expressed it, a *façon de parler* [a figurative expression], which makes redundant a more encompassing mode of expression. For example, the proposition: "if the positive number *n* becomes infinitely large, then the quotient $\frac{1}{n}$ becomes infinitely small" requires a more expanded and precise definition: "the value of the coefficient $\frac{1}{n}$ (*n* positive) can thus be brought arbitrarily close to Zero, when the number *n* is limited by sufficiently large values". In this sense one speaks about the improper or potential infinite. In sharp and clear contrast to this the alternative formulation views the set (as well as the ordering scheme determined by it), as completed and closed, in itself a firm infinity, insofar as it encompasses infinitely many precisely defined elements (the natural numbers), nothing more and nothing less. What is here present is therefore a proper or actual infinity, which is a cognitive entity brought to consciousness in a pure, non-contradictory unified act.[4]

Fraenkel elaborates this point further in a footnote in which he opposes the constantly recurring and apparently ineradicable misunderstanding, namely "that the 'infinity' of the set of all natural numbers has nothing to do with the alleged becoming infinite of

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3 What Fraenkel explains here is similar to the two directions of the whole-parts relation found in B Fr. 3 of Zeno discussed earlier.

4 "[J]ene Größen können über jeden noch so großen endlichen Betrag hinauswachsen oder sich unbegrenzt der Null annähern, ohne daß dieser Zu- oder Abnahme bestimmte Grenzen gesetzt sind. In jedem Stadium des Prozesses, wie immer und wie weit er auch durchgeführt werde, haben die Größen jedoch bestimmte endliche bzw. von Null verschiedene Werte. Es handelt sich also bei diesem Gebrauch des Begriffs 'Unendlich' nur, wie sich GAUSS ausdrückte (vgl. oben S. 1), um eine *façon de parler*, die eine umständlichere Ausdrucksweise entbehrlich macht; z. B. würde der Satz: 'wird die positive Zahl n unendlichgroß, so wird der Ouotient $\frac{1}{\mathrm{n}}$ unendlichklein' ausführlicher und schärfer so zu fassen sein: 'der Wert des Quotienten $\frac{1}{n}$ (*n* positiv) kann dadurch der Null beliebig nahegebracht werden, daß man die Zahl *n* auf hin[7/8]reichend große Werte beschränkt'. Man spricht in diesem Sinne vom uneigentlichen oder potentiellen Unendlich. In scharfem und deutlichem Gegensatz hierzu ist die im vorigen Absatz betrachtete Menge (wie auch das durch sie bestimmte Ordnungsschema) ein fertiges, abgeschlossenes, in sich festes Unendliches, insofern als sie unendlichviele genau definierte Elemente (die natürlichen Zahlen) umfaßt, keines mehr und keines weniger. Hier liegt also ein eigentliches oder aktuales Unendlich vor, das als reines, in einem einheitlichen Akt zum Bewußtsein gebrachtes Gedankending nichts Widerspruchsvolles in sich zu bergen scheint. Das nämliche gilt für die drei folgenden Beispiele" (Fraenkel 1928:7-8).

its elements, i.e., the integers, which are much rather all finite. The set of fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ and so on, in the same way represents an infinity as the set observed above."[5]

This explanation contains various key terms that are interrelated, such as the distinction between the potential and the actual infinite, the difference between "becoming infinite" and "being finite," the difference between varying approximations and a firm and completed whole as well as a unified multiplicity. What is striking is that the concept of infinity brings to expression the interconnectedness of all these key terms. In particular the notion of infinity comes into its own in the idea of a *set*.

11. SET THEORY: A NEW FOUNDATION FOR MATHEMATICS

Cantor's set theory (*Mengenlehre*) indeed achieved the status of being *foundational* for mathematics as a whole. He was convinced "that Set Theory deals with the actual infinite" (Robinson 1967:39). By using the completed infinite Cantor, in 1874, proved that the set of all real numbers cannot be enumerated in the manner of the set of all natural numbers, i.e. that real numbers are non-denumerable. But exactly in this proof, which uses the actual infinite, Herbert Meschkowski (1972b:25) observes the "foundation of set theory."

For this reason it is important to consider the way in which Georg Cantor combined actual infinity and the concept of a set, keeping in mind that Lorenzen holds that "one can say that in arithmetic no motive is found for the introduction of actual infinity."[6]

12. CANTOR'S DEFINITION OF A SET

Cantor is explicit about the key terms involved in his definition of a set:

> Under a 'set' we understand every collection *M* of specific, properly distinct objects *m* of our intuition or our thought (which are designated as the 'elements' of *M*) into a whole[7]

Remember for a moment that Fraenkel alluded to the idea of "a unified multiplicity." Clearly the term "multiplicity" is equivalent to Cantor's "wohlunterschiedenen Objekten" while the term "unified" is equivalent to what Cantor has in mind in holding that the multiplicity is collected "into a whole." The two central points of this definition of a set provided by Cantor are therefore given in

(a) a *multiplicity* of properly distinct objects/elements ("wohlunterschiedenen Objekten ... 'Elemente'") and

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5 "Hier und da immer wieder auftretenden, anscheinend unausrottbaren Mißverständnissen gegenüber werde scharf hervorgehoben, daß dieses 'Unendlich' der Menge aller ganzen Zahlen nichts zu tun hat mit einem vermeintlichen Unendlichwerden der Elemente, d. h. der ganzen Zahlen, die vielmehr alle endlich sind. Die Menge der Brüche $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ usw. stellt ja im gleichen Sinn wie die oben betrachtete Menge ein Unendlich dar!" (Fraenkel 1928:8).

6 "In der Arithmetik – so wird man zusammenfassend sagen können – liegt kein Motiv zur Einführung von Aktual-Unendlichem vor" (Lorenzen 1972:159).

7 "Unter einer 'Menge' verstehen wir jede Zusammenfassung *M* von bestimmten wohlunterschiedenen Objekten *m* unserer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) zu einem Ganzen" (Cantor 1962:282).

(b) their collection into a *whole*.

Point (a) refers to (a quantitative) unity and multiplicity [the one and the any: the *numerical*] and (b) refers to the whole-parts relation [the *spatial*]. Interestingly Russell considers the whole-parts relation as *primitive*: "The relation of whole and part is, it would seem, an indefinable and ultimate relation" (Russell 1956:138).[8]

The explanation given by Cantor captures all these elements.

13. CANTOR DISTINGUISHING BETWEEN THE POTENTIAL INFINITE AND THE ACTUAL INFINITE

It reads as follows:

> I The potential infinite is preferably expressed when an undetermined, variable finite value appears, which either grows beyond all finite limits . . . or decreases below all finite limits in smallness . . ."
> II The actual infinite by contrast is a quantum which on the one hand is not variable, but is much rather firm and determined in all its parts, a genuine constant, while at once on the other hand exceeding every similar finite magnitude. As example I present the totality (*Gesammtheit*), the sum (*Inbegriff*) of all finite positive integers; this set is something in itself and it constitutes, fully disregarding the natural succession of the numbers belonging to it, a quantity that is firm and determined in all its parts.[9]

14. SET THEORY: THE WHOLE AND ITS PARTS GIVEN AT ONCE

Within Zermelo-Fraenkel axiomatic set theory [ZF] this distinction is introduced by commencing with the primitive set theoretic notion of the *membership relation* (∈ – Fraenkel et.al. 1973:22-23).[10] In following Frege a distinction is drawn between ∈ en ⊆.

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8 Aristotle already realized that truly primitive terms cannot be defined since it will terminate in an *infinite regress*: "For it is impossible that there should be demonstration of absolutely everything (there would be an infinite regress so that there would still be no demonstration" …) (Aristotle, *Metaph*. 1006a8-10; 2001:737). Russell is famous for his discovery of the inconsistency of Cantor's set concept. He considered the set C having as elements [members in terms of ZF] those sets that do not contain themselves as elements and then showed that in this case C is an element of C if and only if C is not an element of C.

9 "I. Das P.-U. wird vorzugsweise dort ausgesagt, wo eine unbestimmte, veränderliche endliche Größe vorkommt, die entweder über alle endlichen Grenzen hinaus wächst (unter diesem Bilde denken wir uns z. B. die sogenannte Zeit, von einem bestimmten Anfangsmomente an gezählt) oder unter jede endliche Grenze der Kleinheit abnimmt (was z. B. die legitime Vorstellung eines sogenannten Differentials ist); allgemeiner spreche ich von einem P.-U. überall da, wo eine unbestimmte Größe in Betracht kommt, die unzählig vieler Bestimmungen fähig ist.
II. Unter einem A.-U. ist dagegen ein Quantum zu verstehen, das einerseits nicht veränderlich, sondern vielmehr in allen seinen Teilen fest und bestimmt, eine richtige Konstante ist, zugleich aber andrerseits jede endliche Größe derselben Art an Größe übertrüft. Als Beispiel führe ich die Gesamtheit, den Inbegriff aller endlichen ganzen positiven Zahlen an; diese Menge ist ein Ding für sich und bildet, ganz abgesehen von der natürlichen Folge der dazu gehörigen Zahlen, ein in allen Teilen festes, bestimmtes Quantum" (Cantor 1932:401).

10 See also Bernays and Fraenkel, 1958:5 – where they accept "an (undefined) primitive relation, the membership relation." Interestingly, as remarked by Tait, both Cantor and Dedekind avoids the null set

Fraenkel et.al. note that "while a set always *includes* itself and its subsets, it *contains,* in general, neither itself nor its subsets" (Fraenkel et.al. 1973:27).

And keep in mind what they explain on page 23:

> Throughout the present chapter we shall mean by element an object which is a member of some object. In **ZF** the term 'element' is synonymous with the term 'object', yet we shall prefer to use the former so as to facilitate the comparison of **ZF** with those systems of set theory discussed in §7 in which not all objects are elements. Let us refer to those elements which have members as ***sets,*** and to those elements which have no members as individuals.

In his description, on the same page, of the nature of 'Teilmengen' ('subsets') Cantor employs the phrase "at once" ('zugleich') – showing that the spatial meaning of simultaneity, of the "at once," is entailed in ZF. Yet the *Axiom of Infinity* still merely reflects the numerical (time) order of succession. It is only when it is combined with the *Power Set Axiom* that the original numerical meaning of infinity, highlighted in an endless succession, is altered through the idea of an *infinite totality* (*whole*), evident in the presence *at once* of all the members. The two key elements in Cantor's circumscription of a set are still maintained – a given (successively) infinite multiplicity of members ('elements') united into a whole (*Ganzen*). In ZF the term 'set' is undefined even though its implicit spatial connotation, expressed in the whole-parts relation, is still present, for without this implicit assumption the *Power Set Axiom* cannot explore the distinction between a set and its *subsets*.

15. SETS: THEIR DEPENDENCE UPON SPATIAL FEATURES

In the light of our preceding analysis it should therefore not surprise us that Gödel discerns something "quasi-spatial" in the term *set*. The mathematician Wang remarks: "I am not sure whether he would say the same thing of numbers" (Wang 1988:202). Furthermore, Gödel points out that the term 'set' is 'indefinable':

> The operation 'set of x's' (where the variable 'x' ranges over some given kind of objects) cannot be defined satisfactorily (at least not in the present state of knowledge), but can only be paraphrased by other expressions involving again the concept of set, such as: 'multitude of x's', 'combination of any number of x's', 'part of the totality of x's', where a 'multitude' ('combination', 'part') is conceived as something that exists in itself, no matter whether we can define it in a finite number of words (so that random sets are not excluded) (Gödel 1964:262).

In the case of truly primitive (and indefinable) terms there are only two epistemic options: (i) use a term derived from another domain with its own distinct primitive terms, or (ii) employ synonymous terms. The primary question, however, is still whether or not the term 'set'

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since "no whole has zero parts" (Tait 2005:253). In this Chapter Tait discusses "Cantor's *Grundlagen* and the Paradoxes of Set Theory" (pp.252-275).

represents a (purely) arithmetical concept? It may not be a geometrical concept, but this does not rule out what we have observed thus far, namely that the set concept does involve terms derived from the domain of space. What is striking to note is how many terms with spatial connotations surface in Gödel's account of the indefinability of the term ‘set’. Consider the following instances: ‘part of the totality of x's’; ‘combination,’ ‘part,’; and “as something that exists in itself.”

16. RUSSELL: DISCRETENESS ENTAILS PROGRESSIONS

It should be noted that Bertrand Russell did realize that a multiplicity reflects the primitive meaning of number and numerical succession (progression). For this reason “greater and less are undefinable” (Russell, 1956:194; 167). Later on he remarks that “progressions are the very essence of discreteness” (Russell, 1956:299). In an earlier context he also criticizes Bolzano for not distinguishing the “many from the whole which they form” (Russell, 1956:70).

17. HILBERT EXPLAINING THE TWOFOLD NATURE OF THE INFINITE

Compare this outcome with the way in which Hilbert explains Cantor's approach in 1925:

> Someone who wished to characterize briefly the new conception of the infinite which Cantor introduced might say that in analysis we deal with the infinitely large and the infinitely small only as limiting concepts, as something becoming, happening, i.e., with the potential infinite. But this is not the true infinite. We meet the true infinite when we regard the totality of numbers 1, 2, 3, 4, ... itself as a completed unity, or when we regard the points of an interval as a totality of things which exists all at once. This kind of infinity is known as actual infinity.

First of all it is clear that the role of variables in mathematical analysis is related to the potential infinite and employed on the basis of the (mathematical) limit concept. However, Cantor holds that the use of variables presupposes a “domain of variability” – and it is this *domain* [*Gebiet*] that is investigated by him in his theory of sets (*Mengenlehre*) (Cantor 1967:250). Therefore the “true infinite” does not contain something variable, showing that he rather opts for a static spatial characterization of the actual infinite.

18. ACTUAL INFINITY AS THE BASIS OF THE POTENTIAL INFINITE

Moreover, in a letter from June 1886 Cantor argues in particular that the theory of irrational numbers cannot be maintained apart from having its foundation, in one form or another, in an *actual infinity*. His general claim reads: “So every potential infinite, should it be applied in an exact mathematical way, presupposes an actual infinity.”[11] Hilbert explains the idea of the actual infinite with reference to counting the numbers 1, 2, 3, . . . : “we can regard the objects thus enumerated as an infinite set existing all at once in a particular order” [“eine … fertige unendliche Menge” – see also the translation in Benacerraf 1964:140] (Hilbert 1925:169).

[11] “So setzt jedes potenzial Unendliche, soll es streng mathematisch verwendbar sein, ein actual Unendliches voraus” (quoted by Meschkowski 1967:250).

19. REAL NUMBERS: BEING PRESENT AT ONCE

More recently Lorenzen describes what Cantor had in mind with reference to the real numbers: "One rather imagines the real numbers as all being present at once – even every real number is imagined as an infinite decimal fraction as if the infinitely many numbers exist all at once."[12]

Lorenzen also points out that the idea of the set of real numbers is apparently motivated by geometry because we "continue to speak about the arithmetical continuum" ["Man spricht daher auch stets vom arithmetischen Kontinuum"] (Lorenzen 1972:162). He is even more explicit when he states that owing to the actual infinite entailed in the concept of a real number its descent from geometry could still be recognized. In other words, the connection between the real numbers and space [geometry] derives from the presence of the actual infinite!

Bernays, the co-worker of David Hilbert, distances himself from the philosophy of the "As If" ["Philosophie des Als Ob"] of Vaihinger, for he rejects anything contradictory in his own approach (cf. Bernays 1976:60 and Vaihinger 1949:61–4).

20. BERNAYS: ARITHMETIZING MATHEMATICS IS AN "ARBITRARY THESIS"

Add to this the critical attitude of Bernays regarding the alleged arithmetization of mathematics, which he opposes as an "arbitrary thesis," since it has forgotton that "the idea of the continuum is a geometric idea which is expressed in an arithmetical language by analysis" – originally "the idea of the continuum is a spatial idea" (Bernays 1976:188, 74).

Bernays criticizes intuitionistic mathematics for lacking an insight into the totality character of the continuum. He holds that this feature resists a complete arithmetization of the continuum.[13]

21. BERNAYS AND BROUWER: EXPLORING TWO FEATURES OF WHOLENESS

Something remarkable occurs here. Both Bernays and Brouwer make an appeal to the original spatial feature of wholeness but lift out two distinct sides of it. Bernays mentions the *totality character* of the continuum whereas Brouwer focuses on the *whole-parts* relation. Weyl remarks that Brouwer's view of the *continuum* is in "agreement with intuition" for he "sees the essence of the continuum not in the relation of the element to the set, but in that of the part to the whole" (Weyl 1966:74). In line with the long-standing conviction of Aristotle Weyl also emphasizes the *infinite divisibility* of a continuum: "It rather belongs to the essence of the continuum that each of its parts are infinitely divisible" (Weyl 1921:77). Weyl is also critical of the idea of a *continuum of points*. The concept of an 'environment' is still required

[12] "Man stellt sich vielmehr die reelle Zahlen als alle auf einmal wirklich vorhande vor – es wird sogar jede reelle Zahle als unendlicher Dezimalbruch selbst schon so vorgestellt, als ob die unendlich vielen Ziffern alle auf einmal existierten" (Lorenzen 1972:163).

[13] "daß die intuitionistische Vorstellung nicht jenen Charakter der Geschlossenheit besitzt, der zweifellos zur geometrischen Vorstellung des Kontinuums gehört. Und es ist auch dieser Charakter, der einer vollkommenen Arithmetisierung des Kontinuums entgegensteht" (Bernays 1976:74).

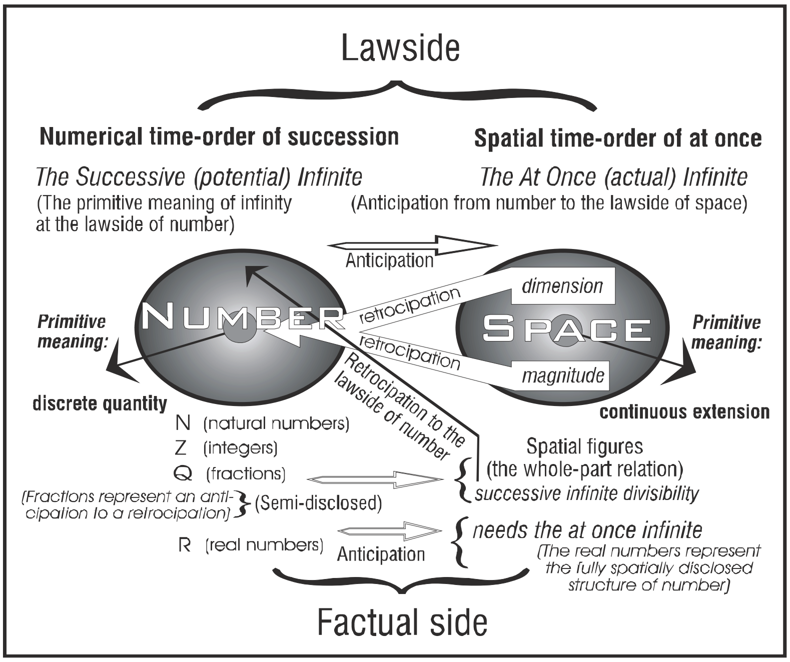
to safe-guard continuity: “To render a continuous coherence of points analysis has up to today (since it decomposed the continuum into a set of isolated points) found refuge in the concept of environment.”[14] Laugwitz also consider this critical stance: “The problem of the continuum appears to be decided in favour of the atomistic conception: it is nothing but a set of (isolated) points, upon which a set theoretic auxiliary construction (environments, open sets) then impregnates a ‘topology’ such that one can once more speak about ‘continuity’ ”[15]

22. PRIMITIVES AND THEIR COHERENCE

Yourgrau explains that Gödel “insisted that to know the primitive concepts, one must not only understand their relationships to the other primitives but must grasp them on their own, by a kind of ‘intuition’ ”(Yourgrau, 2005:169). To this Yourgrau adds another view of Gödel, namely that “the fundamental concepts are primitive and their meaning is not exhausted by their relationships to other concepts” (Yourgrau 2005:170).

Figure 1

Infinity: embedded in the connection between number and space



The mutual coherence and irreducibility of number and space

Figure 1 exemplifies this view of Gödel. It first of all captures the *primitive* meaning of the aspects of number and space, namely “discrete quantity” and “continuous extension.” In addition it highlights their mutual coherence, expressed in retrocipations and anticipations

14 Um den stetigen Zusammenhang der Punkte wiederzugeben, nahm die bisherige Analysis, da sie ja das Kontinuum in eine Menge isolierter Punkte zerschlagen hatte, ihre Zuflucht zu dem *Umgebungsbegriff*. (Weyl 1921:77).

15 Das Problem des Kontinuums scheint zugunsten der atomistischen Auffassung entschieden u sein: es ist nichts anderes al seine Menge von (isolierten) Punkten, der mit einer Mengentheoretischen Hilfkonstruktion (Umgebungen, offene Mengen) dann eine ‘Topologie’ aufgesprägt wird, in der man wieder von einder ‘Stetigkeit’ reden kann” (Laugwitz 1997:266).

(backward and forwards pointing *analogies*). On the law-side of these aspects the difference between the successive infinite (potential infinite) and the at once infinite (actual infinite) presents itself and at once serves what is given at the factual side of these aspects, such as different kinds of number and the whole-parts structure of spatial figures. In the system of real numbers the fully spatially disclosed structure of number is found, imitating the original spatial meaning of continuity. Integers imitate the feature of spatial wholeness, while fractions imitate the whole-parts relation at the factual side of the spatial aspect.

Our preceding analysis has now sufficiently shown that 'infinity' is not a purely arithmetical notion even though its most basic form, given in the *successive infinite*,[16] is indeed 'purely' arithmetical. However, the spatially deepened meaning of the at once infinite anticipates key features of the spatial aspect, namely the spatial time order of at once (simultaneity) and the factual spatial whole-parts relation.[17]

In the footsteps of Weierstrass, Cantor and Dedekind, the leading mathematicians of the 20th century adhered to the idea that the set of real numbers is continuous and therefore constitutes the *continuum*. Yet Laugwitz correctly points out that in Cantor's definition of a set "the discrete rules" (Laugwitz 1986:10 – "das Diskrete herrscht"). Each number has distinctive features, whereas spatial points and lines do not have distinct qualities. Yet the idea of infinite totalities exceeds the realm of a discrete multiplicity for it imitates (anticipates) the spatial feature of being a *totality* or a *whole*. It implicitly also makes an appeal to the spatial time order of simultaneity (at once), because all the parts of a spatial whole must be present *at once*.

23. INFINITE TOTALITIES: "AS IF" THEY EXIST

In passing we may mention the fact that Robinson, known for his contribution to non-standard analysis, disqualifies infinite totalities as non-existent and literally *meaningless.* Nonetheless he holds that "we should continue the business of mathematics 'as usual', i.e., we should act *as if* infinite totalities really existed." To this he adds the remark that infinite totalities could be treated as well-founded fictions (*fictiones bene fundatae*) (Laugwit 1986:234). This view is partly similar to the view of Bernays mentioned earlier.

16 Immanuel Kant acknowledges the infinite divisibility of continuity (Kant 1787:206) and in one instance (in his *Kritik der reinen Vernunft*) he speaks about the successive infinite (*sukzessivunendlich*) (Kant 1787:552). He stumbled upon the idea of the at once infinite in a negative way, for he holds that the successively infinite series of divisions is never complete (*Ganz*) and therefore does not constitute an infinite set: "Da dieser Regressus nun unendlich ist, so sind zwar alle Glieder (Teile), zu denen er gelangt, in dem angegebenen Ganzen als Aggregate enthalten, aber nicht die ganze reihe der Teilung, welche sukzessivunendlich und niemals ganz ist, folglich keine unendliche Menge, und keine Zusammennehmung derselben in einem Ganzen darstellen kann" (Kant 1787:552).

17 When Bell explains that "smooth infinitesimal analysis" [SIA] provides "an image of the world in which the continuous is an autonomous notion, not explicable in terms of the discrete" (Bell 2006:18), his view approximates what is intended in Figure 1. Although Fraenkel believes that Cantor aimed at a fusion of arithmetic and geometry in which equal justice will be done to both ["A 'wahre Fusion von Arithmetik und Geometrie' durch die Mengenlehre zu schaffen, innerhalb deren beide ihre gleichberechtigten Plätze finden sollten, war CANTORS bewußtes Streben" (Fraenkel 1928:391-392)], it should be remembered that Cantor argued on the basis of the real numbers for a possibly general purey arithmetical concept of a point-continuum ["Somit bleibt mir nichts anderes übrig, als mit Hilfe der in § 9 definierten reellen Zahlbegriffe einen möglichst allgemeinen rein arithmetischen Begriff eines Punktkontinuums zu versuchen"] (Cantor 1932:192).

The difference between them is given in the sensitivity which Bernays reveals for basic ontological distinctions. He acknowledges two kinds of factuality, such as aspectual (modal functional) subjects (number and spatial figures that are factually subjected to their corresponding numerical and spatial laws) and typical subjects (factual entities, such as atoms and molecules, material things, plants, animals and humans). Where he makes an appeal to the intuitive, the theoretical and the experimental he implicitly touches upon the just-mentioned distinctions: [“dem intuitive, dem theoretischen und dem experimentellen”] (Bernays, 1976:108). In his discussion of Wittgenstein it is noteworthy that Bernays rejects the view of those who merely acknowledge one kind of factuality, that which is concrete: “It appears that only a pre-conceived philosophical view determines this requirement, the view namely, according to which solely one kind of factuality can exist, that of concrete reality.”[18]

24. WEYL: THE FALL AND ORIGINAL SIN OF SET-THEORY

Weyl accuses the employment of the idea of infinite totalities as the basic sin of mathematics: “This is the Fall and original sin of set-theory, for which it is justly punished by the antinomies” (Weyl 1946:10 – see note 7 above). This objection implicitly denies the *ontic status* of the (modal) aspects of number and space. In the absence of a theory of functional modalities (modes of being), Gödel took recourse to the idea of ‘semiperceptions’ in order to account for mathematical concepts. He proposes the notion of “semiperceptions” in connection with “mathematical objects.” Distinct from the physical existence of things Gödel mentions data of a second kind which are open to “semiperceptions.” But the data belonging to this second kind “cannot be associated with actions of certain things upon our sense organs” (quoted by Wang 1988:304). Gödel argues:

> It by no means follows, however, [that they] are something purely subjective as Kant says. Rather they, too, may represent ‘an aspect of objective reality’ but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality (quoted by Wang 1988:304).

25. GÖDEL AND WANG CONTEMPLATE “ONTIC” (“OBJECTIVE”) “ASPECTS OF REALITY”

Wang declares that he is “inclined to agree with Gödel,” even though he does “not know how to elaborate his assertions” (Wang 1988:304). He explains that he “used to be troubled by the association of objective existence with having a fixed ‘residence’ in spacetime,” but he now feels “that ‘an aspect of objective reality’ can exist (and be ‘perceived by semiperceptions’) without its occupying a location in spacetime in the way physical objects do” (Wang 1988:304). Clearly, Gödel and Wang contemplate the “reality” of “ontic” (designated by them as: “objective”) “aspects of reality” which are not like “concrete entities” occupying “a location in spacetime.”

26. CONCLUDING REMARK

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[18] “Es scheint, daß nur eine vorgefaste philosophische Ansicht dieses Erfordernis bestimmt, die Ansicht nämlich, daß es nur eine Art von Tatsächlichkeit geben könne, diejenige der konkreten Wirklichkeit” (Bernays 1976:122).

In conclusion we may therefore say that the original and primitive concept of infinity is given in the *successive infinite* which is purely arithmetical in nature. However, this kind of infinity is deepened and disclosed in the *at once infinite* which guides theoretical thinking to view any successively infinite multiplicity as being given *at once*, without any before and after. Just like all the parts of a spatial figure must be present at once, simultaneously, for otherwise the figure does exist. The three sides of a triangle cannot exist in succession, they must be present at once.

Consider the following example.

> The sequence row of natural numbers, commencing with 11, 1 2, 13, . . . could be extended indefinitely because it is an instance of the *successive infinite*. Suppose now that we consider the fractions $\frac{1}{1}, \frac{1}{2}, \frac{1}{3},\ \ldots$ and map them on the corresponding points within the closed interval [0,1]. Then, by employing our spatial intuition of simultaneity, we can view the initial successively infinite sequence of numbers, 11, 1 2, 13, . . . as if they are all present *at once* as an *infinite totality* – represented by the points correlated with the mediating fractions $\frac{1}{1}, \frac{1}{2}, \frac{1}{3},\ \ldots$ (through a one-one mapping).

Alternatively ask the question whether or not 0.999 . . . is equal to 1. From the perspective of the successive infinite it is *not* equal to one, but in terms of the at once infinite it *is*.[19]

The attempt launched by Adolf Grünbaum to advance a consistent conception of the extended linear continuum as an aggregate of unextended elements is crucially dependent upon the non-denumerability of the real numbers (see Grünbaum 1952:302). Yet proving the non-denumerability of the real numbers fully depends upon the employment of the at once infinite which, in turn, presupposes the spatial whole-parts relation and the at once infinite (see Fraenkel 1928:239, note 1).

In other words, the *at once infinite* presupposes the irreducible, unique nature of the spatial aspect and cannot be used subsequently to reduce space to number, i.e., to arithmetize mathematics. The reductionist attempt is *antinomical* and implies the following *contradiction*: space can be reduced to number if and only if it cannot be reduced to number, i.e., if and only if the *at once infinite* is used, which presupposes the irreducibility of the spatial aspect![20]

The impossibility to articulate the nature of the at once infinite without (implicitly or explicitly) exploring key elements of space therefore uproots the conclusion reached by Buckley in his work on the continuity debate. He writes: “We are also guaranteed that mathematics can stand on arithmetic feet, without constant reference to geometry” (Buckley

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19 Weierstrass, Cantor and Dedekind defined real numbers by employing the actual infinite – respectively using an actually infinite static domain of numerical values (see Boyer 1959:286), employing the idea of “fundamental sequences” (*Fundamentalreihe* – see Cantor 1932:186) and by introducing a Dedekind *cut* (see Dedekind 1872:16-17 - § 6).

20 This issue is treated in more detail in Strauss 2011.

2012:164). Unfortunately employing the at once infinite (the actual infinite) does need a spatial 'leg' – found in the *whole-parts relation* and the spatial time order of *at once*. Remember what we have mentioned in note 5 above, namely that Lorenzen aptly pointed out that arithmetic provides no motif for introducing the *at once* infinite (Lorenzen 1972:159).

LITERATURE

Aristotle.2001. *TheBasicWorksof Aristotle*.Edited byRichard McKeon withanIntroduction by C.D.C. Reeve. (Originally published by Random House in 1941). New York: The Modern Library.

Becker, O. 1964. *Grundlagen der Mathematik in geschichtlicher Entwicklung*, München: Alber.

Bell, J.L. 2006. *The continuous and the infinitesimal in mathematics and philosophy*. Milano: Polimetrica, Corso.

Benacerraf, P. and Putnam, H. (red.) 1964. *Philosophy of Mathematics*, *Selected Readings*. Oxford: Basil Blackwell.

Bernays,P.1976. *Abhandlungen zur Philosophie der Mathematik*. Darmstadt: Wissenschaftliche Buchgesellschaft.

Bolzano, B. 1851. *Paradoxien es Unendlichen*. Leipzig: Reclam.

Boyer, C.B. 1959. *A History of the Calculus and its Conceptual Development*. New York: Dover Publications Inc.

Buckley, B.L. 2012. *The Continuity Debate*: *Dedekind*, *Cantor*, *du Bois-Reymond*, and *Pierce on Continuity and Infinitesimals*. Boston (Massachusetts): Docent Press.

Cantor, G. 1962. *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*. Hildesheim: Georg Olms Verlagsbuchhandlung.

Dedekind, R. 1872. *Stetigkeit und Irrationale Zahlen*. Studienausgabe. Unaltered copy of the seventh edition. Braunschweig: E. Hunold.

Diels, H. and Kranz, W. 1960. *Die Fragmente der Vorsokratiker*. Vols. I-III. Berlin: Weidmannsche Verlagsbuchhandlung.

Fraenkel, A. 1928. *Einleitung in die Mengenlehre*. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsebiete, BAND IX, Gemeinsam mit Blaschke W. Born, M. und Runge W., Herausgegeben von R. Courant, Dritte Umgearbeitete und stark erweiterte Auflage [Third revised and expanded edition], Göttingen: Springer.

Gödel, K. 1964. What is Cantor's continuum problem? In Benacerraf en Putnam (reds.) 1964. In: Benacerraf, P. en H. Putnam (red.) 1964. *Philosophy of mathematics*, *selected readings*. Oxford: Basil Blackwell.

Gödel, K. 1964. What is Cantor's continuum problem? In: Benacerraf & Putnam, 1964:258-273.

Grünbaum, A. 1952. A consistent conception of the extended linear continuum as an aggregate of unextended elements. *Philosophy of Science*. 19(2):288-306.

Hasse, H. and Scholz, H. 1928. Die Grundlagenkrisis der griechischen Mathematik. *Kant-Studien*, Vol. 33, 5-34.

Hilbert, D. 1925. Über das Unendliche, Mathematische Annalen, Vol.95, 1925: 161-190.

Kant, I. 1787. *Kritik der reinen Vernunft*, Second Edition (references to CPR B). Hamburg: Felix Meiner edition (1956).

Laugwitz, D. 1986. *Zahlen und Kontinuum*, *Einführung in die Infinitesimalmathematik*. Darmstadt: Wissenschaftliche Buchgesellschaft.

Laugwitz, D. 1997. Mathematische Modelle zum Kontinuum und zur Kontinuität. *Philosophia Naturalis*, 34:265-313.

Lorenzen, P. 1972. Methodisches Denken. In: Meschkowski (Ausszüge) 1972:157-178.

Meschkowski, H. 1967. *Probleme des Unendlichen*. *Werk und Leben Georg Cantors*. Braunschweig: Friedr. Vieweg & Sohn.

Meschkowski, H. (Ed.) 1972. *Grundlagen der Mathematik*. Darmstadt: Wissenschaftliche Buchgesellschaft.

Meschkowski, H. (Ed.) 1972b.Der Beitrag der Mengenlahre zurGrundlagenforschung. In: Meschkowski, 1972:21-55.

Russell, B. 1956. *The Principles of Mathematics*. London: George Allen & Unwin. (First published in 1903, Second edition 1937, Seventh edition 1956).

Schilder, K. 1948. *Christus en Cultuur*. Franeker: Wever.

Strauss, D.F.M. 2011. (Oor)aftelbaarheid. *Koers* 26(4):637-659.

Tait, W. 2005. *The Provenance of Pure Reason*. *Essays in the Philosophy of Mathematics and its History*. Oxford: University Press.

Thesleff, H. 1970. The Pythagoreans in the light and shadows of recent research. In: *Mysticism*, Based on Papers read at the Symposium on Mysticism held at Abo on the 7th-9th September 1968 (Scripta Instituti Donneriani Aboensis, V). (edited by Hartman, S.S. en Edsman, C.M.). Stockholm: Almqvist & Wicksell (160 pp).

Weyl, H. 1921. Über die neue Grundlagenkrise der Mathematik. Berlin: *Mathematische Zeitschrift*, 10:39-79.

Weyl, H. 1946. Mathematics and Logic. *American Mathematical Monthly*, Vol. 53:1-13.

Weyl, H. 1966. *Philosophie der Mathematik und Naturwissenschaft*, 3rd revised and expanded edition. Vienna: R. Oldenburg.

Yourgrau, P. 2005. *A World Without Time*. *The forgotten Legacy of Gödel and Einstein*. London: Penguin Books.